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The Inverse of Transformation Method in Stochastic Mechanical Structure

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Abstract: In this paper, a proposed technique for the inverse of the transformation method is presented. The direct transformation technique, recently developed by the authors [9], evaluateSeptember 16, 2007s the probability density function (p.d.f) -in closed form- of the response (i.e. displacement) of stochastic mechanical system; where the probability density function of the input (i.e. Young's modulus, load…) is known. The inverse technique, presented in this article, uses the direct transformation method but instead of evaluating the p.d.f of the response, we evaluate the p.d.f of the input by supposing the p.d.f of the output. This method is very powerful in probabilistic analysis of a stochastic structure and for the industry during the statistical analysis and design process of a new system.

Keywords: Finite element method, Probabilistic methods, Sampling, Sensitivity; Simulation, Transformation method.

INTRODUCTION

The Stochastic Finite Element Method (SFEM) represents a new approach to solve mechanical systems with stochastic characteristics. The SFEM is based on the deterministic Finite Element Method in which some variables related to the structural state (variables involved in the stiffness matrix) and to the applied actions (involved in the load vector) are uncertain. In other words, the SFEM tries to look for the stochastic properties of the mechanical response.

The problem of reliability analysis of a stochastic mechanical system is of central importance in the safety assessment of structures. In a stochastic system, a large number of random variables influence the performance of the system, e.g. Young's modulus, external load… The performance of the system is evaluated by a *best-estimated* code. Suppose performance criterion Y of the system depends on the inputs variables $X_1, X_2, \ldots, X_n$. Although the actual response Y is a function of the input variables, i.e., $Y=g(X_1, X_2, \ldots, X_n)$.

In order to get information about the uncertainty of Y, a number of code runs have to be performed. For each of these calculation runs, all identified uncertain parameters are varied simultaneously.

According to the exploitation of the result of these studies, the uncertainty on the response can be evaluated either in the form of uncertainty range, or in the form of a probability density function (*pdf*).

In this paper, we propose a method named *PTM-FEM* in which the *Probabilistic Transformation Method* (*PTM*) is combined with the deterministic *Finite Element Method* (*FEM*) in order to determine the *pdf* of the input of a stochastic mechanical system with random parameters (excitation, stiffness, …).

Uncertainty Range: A two-sided tolerance interval *[m, M]* of a response Y, for a fractileα ?and a confidence level β is given by:

$$P\{P(m \le Y \le M) \ge \alpha\} \ge \beta$$

Such a relation means that one can affirm, with at the most (1-β) percent of chances of error, that at least α ?percents values of the response Y lie between the values m and M[1]. To calculate the limits m and M, the technique usually used is a method of simulation combined with the formula of Wilks[2].

The advantage of using this technique is that the number of code calculation needed is independent of the number of uncertain parameters. However for reliability evaluation, this method is not very useful because it is difficult, indeed impossible to interpret the two levels of probability (α and β) in term of reliability value for the system.

Probability Density Function and Related Works: The solution of a stochastic mechanical system is completely defined through the evaluation of the probability density function of the response process. This cannot be analytically achieved through most of the available methods and techniques such as Fokker-Planck equation, Wiener-Hermite expansion,

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perturbation methods, stochastic linearization, WHEP technique, decomposition method and stochastic finite element methods. Some exact solutions are available for the mean and standard deviation, not for the *Probability Density Function* (*pdf*), of the solution process[3,4].

The uncertainty evaluation in the form of a *pdf* gives richer information than a confidence interval. Once the *pdf* of the system performance is determined, the statistical analysis of the system can be directly obtained. The following paragraphs describe the various methods available for this evaluation.

Method of Monte-Carlo: The method of Monte-Carlo[5] is not only used to build *pdf*, but also to assess the reliability (as we will explain later) of components or structures (evaluate the sensitivity of parameters). Monte Carlo simulation consists of drawing samples of the basic variables according to their probabilistic characteristics and then feeding them into the performance function. In this way, a sample of response $\{Y_j, j = 1,..,N\}$ is obtained.

The main advantage of the Monte-Carlo method is that this method is not only valid for static, but also for dynamic models and for probabilistic model with continuous or discrete variables. The main drawback of this method is that it requires often a large number of calculations and can be prohibitive when each calculation involves a long and onerous computer time.

Response Surface Method: To avoid the problem of long computer time in the method of Monte-Carlo, it can be interesting to build an approximate mathematical model called response surface[7].

Experiments are conducted with design variables $X_1, X_2,\ldots,X_n$ a sufficient number of times to define the response surface to the level of accuracy desired. Each experiment can be represented by a point with coordinates $x_{1j}, x_{2j},\ldots,x_{nj}$ in an n-dimensional space. At each point, a value of y_i is calculated. The basic response procedure is to approximate by a simple mathematical model, such as an n^{th} order polynomial with undetermined coefficients.

When a response surface has been determined, the system reliability can be easily assessed with Monte Carlo simulation, using the approximate mathematical model, but this response surface must be qualified. The practical problems encountered by the use of the response surface method are pound in the analysis of strongly non-linear phenomena; where it is not obvious to find a family of adequate functions, also such problems appear in the analysis of discontinuous phenomena.

Transformation Methods: The Probabilistic Transformation Method is based on one-to-one mapping between the random output(s) and input(s) where the transformation Jacobian *J* can be computed. The *pdf* of the output(s) is then computed through the known joint *pdf* of the inputs multiplied by the determinant of transformation Jacobean matrix. The one-to-one mapping condition can be relaxed through some mathematical tricks.

This PTM-FEM allows us to express the "exact" *pdf* of the mechanical response[6,9], provided that the transformation Jacobian can be defined. For many cases, the *pdf* of the response can be obtained in a closed-form in terms of the joint distribution of the input random variables.

The idea of PTM is based on the following formula[8]:

$$f_U(u) = f_P(p)\cdot\left|J_{P,U}\right| = f_P(p)\cdot\left|\frac{\partial\varphi^{-1}(u)}{\partial u}\right| \quad (1)$$

Where p is the input parameter, u is the response (solution) and is the inverse transformation, which can be determined either analytically or numerically.

Inverse of Transformation Methods: The advantage of PTM is the analytical evaluation of the pdf of the output when the relationship input-output and the pdf of the input are known. The question now is: if we suppose that the pdf of the output is given, how can we calculate the pdf of the input?. The answer of this question is possible using the inverse of transformation method. The inverse of transformation method is very important for industries during the design process for a new system.

Algorithm:

- Using the Finite Element Method (FEM) to find the relationship between Input-Output.
- Calculate the direct Jacobian of the transformation evaluated in 1.
- Find the pdf of the input using (1).

Application (3-bars Truss): In the second application, we are going to analyze a three-bar truss structure (Figure 1) with random parameters (Young's modulus E or concentrated load P).

FEM Modeling the Three-bar Truss: The element Stiffness matrix, in global axis, is given:

$$\left[K^e\right]^{(i)} = \frac{A^i E^i}{l_i}\begin{bmatrix} \lambda^2 & \lambda\mu & -\lambda^2 & -\lambda\mu \\ \lambda\mu & \mu^2 & -\lambda\mu & -\mu^2 \\ -\lambda^2 & -\lambda\mu & \lambda^2 & \lambda\mu \\ -\lambda\mu & -\mu^2 & \lambda\mu & \mu^2 \end{bmatrix} \quad (2)$$

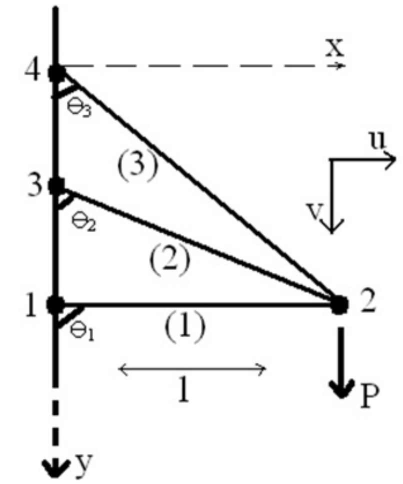


Fig. 1: 3-bar truss structure

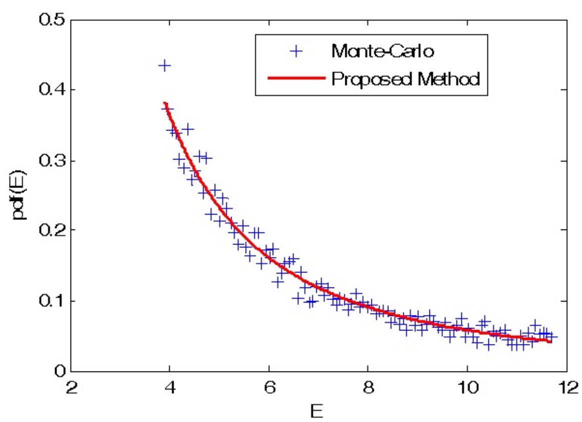


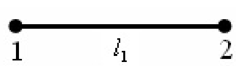
Fig. 2: pdf (E) when   is uniformly distributed

Where:

$$\begin{cases} i = number\ of\ element \\ A = cross\ section \\ E = Young's\ modulus \\ l = length\ of\ bar \\ \lambda = \cos\alpha \\ \mu = \sin\alpha \end{cases}$$

$\alpha = angle\ between\ the\ element\ and\ the\ horizontal$

Element (1): 1-2



$$\begin{cases} \lambda = \cos\alpha_1 = 1 \\ \mu = \sin\alpha_1 = 0 \end{cases}$$

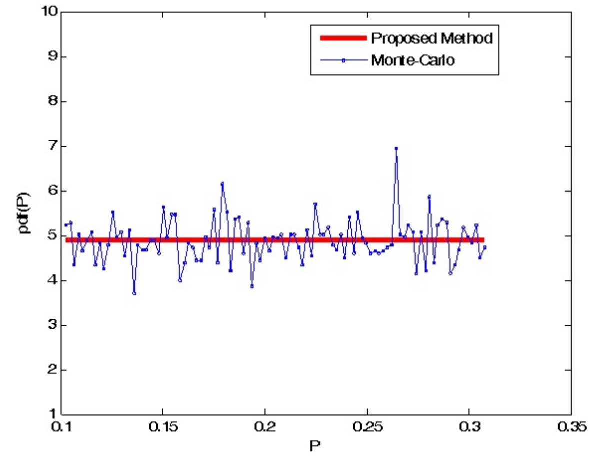


Fig. 3: pdf (P) when   is uniformly distributed

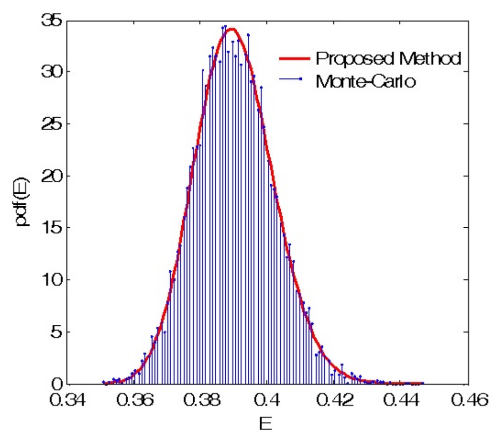


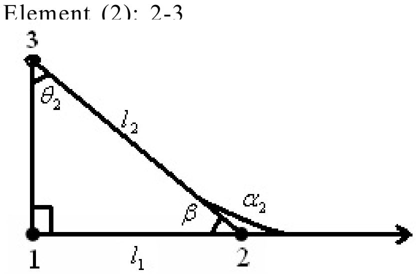
Fig. 4: pdf   when   is normally distributed

The stifness matrix is:

$$\left[K^e\right]^{(1)} = \frac{AE}{l_1}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To simplify, we suppose .

Element (2): 2-3



$$\alpha_2 = \pi - \beta = \frac{\pi}{2} - \theta_2 \qquad (3)$$

$$\lambda = \cos\alpha_2 = \cos\left(\frac{\pi}{2} - \theta_2\right) = \sin\theta_2 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad (4)$$

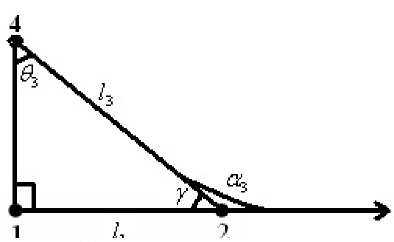
$$\mu = \sin\alpha_2 = \sin\left(\frac{\pi}{2} - \theta_2\right) = \cos\theta_2 = \cos\frac{\pi}{3} = \frac{1}{2} \qquad (5)$$

$$l_2 = \frac{l_1}{\cos\beta} = \frac{2l_1}{\sqrt{3}} \qquad (6)$$

$$\left[K^e\right]^{(2)} = \frac{AE}{l_2}\begin{bmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{3}{4} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & \frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix} \qquad (7)$$

$$\left[K^e\right]^{(2)} = \frac{AE}{l_1}\begin{bmatrix} \frac{3\sqrt{3}}{8} & \frac{3}{8} & -\frac{3\sqrt{3}}{8} & -\frac{3\sqrt{3}}{8} \\ \frac{3}{8} & \frac{\sqrt{3}}{8} & -\frac{3}{8} & -\frac{\sqrt{3}}{8} \\ -\frac{3\sqrt{3}}{8} & -\frac{3}{8} & \frac{3\sqrt{3}}{8} & \frac{3}{8} \\ -\frac{3}{8} & -\frac{\sqrt{3}}{8} & \frac{3}{8} & \frac{\sqrt{3}}{8} \end{bmatrix} \qquad (8)$$

Elements (3): 2-4



$$\alpha_3 = \pi - \gamma = \frac{\pi}{2} - \theta_3 \qquad (9)$$

$$\lambda = \cos\alpha_3 = \cos\left(\frac{\pi}{2} - \theta_3\right) = \sin\theta_3 = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad (10)$$

$$\mu = \sin\alpha_3 = \sin\left(\frac{\pi}{2} - \theta_3\right) = \cos\theta_3 = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad (11)$$

$$l_3 = \frac{l_1}{\cos\gamma} = \frac{2l_1}{\sqrt{2}} \qquad (12)$$

$$\left[K^e\right]^{(3)} = \frac{AE}{l_3}\begin{bmatrix} \frac{2}{4} & \frac{2}{4} & -\frac{2}{4} & -\frac{2}{4} \\ \frac{2}{4} & \frac{2}{4} & -\frac{2}{4} & -\frac{2}{4} \\ -\frac{2}{4} & -\frac{2}{4} & \frac{2}{4} & \frac{2}{4} \\ -\frac{2}{4} & -\frac{2}{4} & \frac{2}{4} & \frac{2}{4} \end{bmatrix} \qquad (13)$$

$$\left[K^e\right]^{(3)} = \frac{AE}{l_1}\begin{bmatrix} \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix} \qquad (14)$$

Using the Assembly Theorem, the global stiffness matrix is (15):

$$[K]^g = \frac{AE}{l_1}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & \frac{8+3\sqrt{3}+2\sqrt{2}}{8} & \frac{3+2\sqrt{2}}{8} & \frac{-3\sqrt{3}}{8} & \frac{-3\sqrt{3}}{8} & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} \\ 0 & 0 & \frac{3+2\sqrt{2}}{8} & \frac{3+2\sqrt{2}}{8} & \frac{-3}{8} & \frac{-\sqrt{3}}{8} & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} \\ 0 & 0 & \frac{-3\sqrt{3}}{8} & \frac{-3}{8} & \frac{3\sqrt{3}}{8} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & \frac{-3}{8} & \frac{-\sqrt{3}}{8} & \frac{3}{8} & \frac{\sqrt{3}}{8} & 0 & 0 \\ 0 & 0 & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & \frac{-\sqrt{2}}{4} & \frac{-\sqrt{2}}{4} & 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \end{bmatrix}$$

Therefore the assembly of three elements leads to the following system

Where:

$$\{F\} = [K]^g . \{U\} \qquad (16)$$

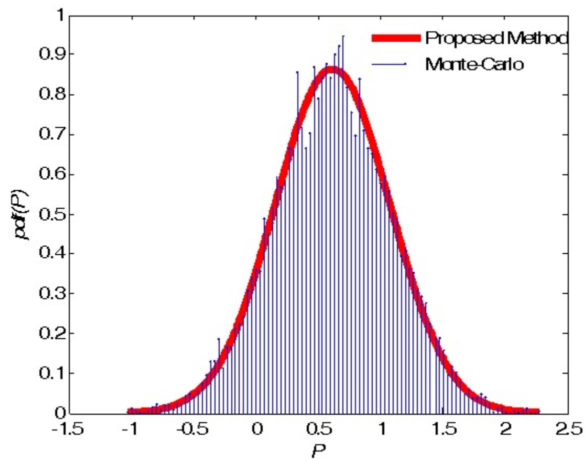


Fig. 5: pdf when is normally distributed

$$\{F\} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ 0 \\ P \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \end{bmatrix} \quad \{U\} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (17)$$

After resolution, the vertical displacement of node 2 is:

$$v_2 = 3.25\frac{Pl_1}{AE} \qquad (18)$$

The previous equation gives the relation between the output v_2 and the input (P, l_l, A, E) Now, the question of the inverse problem is: if we want the distribution of the output v_2 to follow a certain law, what should be the distribution of the input (P, l_l, A, E) In the following cases, the probabilistic analysis of the input is studied where the distribution of the output is supposed to be known.

Probabilistic Study of the Input: Case 1: is uniformly distributed, U(1, 3)

a) Using our technique, the PDF of the Young's modulus is written:

$$PDF(E) = |J|\,PDF(v_2) = \frac{3.25Pl_1}{AE^2}PDF(v_2)$$
$$= \begin{cases} \dfrac{3.25Pl_1}{2AE^2} & if\ \dfrac{3.25Pl_1}{3A} \le E \le \dfrac{3.25Pl_1}{A} \\ 0 & if\,not \end{cases}$$

Numerical values: P=1.2N, l=3m and A=1m2.

b) Using our technique, the PDF of the external load P is:

$$PDF(P) = |J|\,PDF(v_2) = \frac{3.25l_1}{AE}PDF(P)$$
$$= \begin{cases} \dfrac{3.25l_1}{AE} & if\ \dfrac{AE}{3.25l_1} \le P \le \dfrac{3AE}{3.25l_1} \\ 0 & if\,not \end{cases}$$

Numerical values: E=1KN/ m2, l=3m and A=1m2.

Case 2: (N) normally distributed, N(30,0.9)

a) Using our technique, the PDF of the Young's modulus E is:

$$PDF(E) = |J|\,PDF(v_2) = \frac{3.25Pl_1}{AE^2}PDF(v_2)$$
$$= \left\{ \frac{3.25Pl_1}{AE^2}\cdot\frac{1}{0.9\sqrt{2\pi}}.e^{\frac{-(\frac{3.25Pl_1}{AE}-1.2)^2}{2\times 0.9^2}} \right.$$

b) Using our technique, the PDF of the external load P is:

$$PDF(P) = |J|\,PDF(v_2) = \frac{3.25l_1}{AE}PDF(P)$$
$$= \left\{ \frac{3.25l_1}{AE}\cdot\frac{1}{0.9\sqrt{2\pi}}.e^{\frac{-(\frac{3.25Pl_1}{AE}-1.2)^2}{2\times 0.9^2}} \right.$$

Conclusion: In this paper, the inverse problem of the probabilistic transformation method has been developed. It's shown that, if we proposed the distribution of the output variable we can calculate the "exact" probability density function of the input which is very helpful in the design process of a mechanical product. The accuracy of our method has been verified by 10000 simulations of Monte-Carlo.

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