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ORIGINAL ARTICLES

Numerical Solutions of Fredholm and Volterra Integro differential Equations via Optimal Control Approach

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ABSTRACT

In this paper, a numerical method to solve the Fredholm and Volterra integro differential equation is introduced. The method is based on reformulate the Fredholm differential equation to be Fredholm integral equation and hence converts it to optimal control problem; by the same way the Volterra integro differential equation has been solved. The existence and uniqueness of proposed solution are achieved. Numerical results are given at the end of this paper.

Key words: Fredholm integro differential equations, Volterra integro differential equation, optimal control problems

Introduction

The nonlinear integro differential equations arise in the theory of parabolic boundary value problems, some of engineering problems, various mathematical physics and theory of elasticity [Brunner, 2009]. In recent years, several numerical methods for solving linear and nonlinear integro differential equations have been presented [Maleknejad *et. al.*, 2009]. Nonlinear phenomena that appear in many applications in scientific fields can be modeled by nonlinear integro differential equations. Several numerical methods were used such as the successive approximation method, the Adomian decomposition method, the Collocation method, Haar Wavelet method, Wavelet-Galerkin method, Taylor polynomial, the monotone iterative technique and the Tau method [Delves *et. al.*, 1985].These methods often transform an integral or integro differential equation to a linear system of algebraic equations which can be solved by direct or iterative methods. The classical method of successive approximations was introduced in [Pour-Mahmoud, 2005]. In [Nurcan and Sezer, 2010], the authors used Taylor series for solving the following nonlinear Fredholm integro differential equation:

$$\phi^m(x) = f(x) + \sum_{j=0}^{j} \lambda_j \int_a^b k_j(x,t)[\phi(t)]^q dt, \qquad a \le x, t \le b$$

where *a*, *b* and *m* are suitable constants.
In this paper, we are introduced a method to solving the numerical solution of a nonlinear Fredholm integro differential equation(FIDE) and nonlinear Volterra integro differential equations(VIDE), without loss of generality, we take *m=1*, then FIDE in the form:[Hosseini, and Shahmorad, 2003 ; Maleknejad and Hadizadeh, 1997]

$$\phi'(x) = f(x) + \lambda \int_a^b k(x,t)[\phi(t)]^q \, dt, \qquad 0 \le x, t \le 1,$$

where q is a nonnegative integers, $f(x)$ and the kernels $k(x,t)$ are assumed to be in $L^2(R)$ on the interval $0 \le x, t \le 1$.

Legendre wavelets method to a special type was applied in [Yalcinbas and Sezer, 2000], for solving the numerical solution of nonlinear Fredholm integro differential equation of the form:

$$\phi'(x) = f(x) + \lambda \int_a^b k(x,t) G(\phi(t)) dt$$

where $f(x)$ and the kernels $k(x,t)$ are assumed to be in $L^2(R)$ on the interval $0 \le x, t \le 1$. The nonlinear Fredholm integro differential equation is given in [Hosseini and Shahmorad, 2003] as:

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$$\phi'(x) = f(x) + \lambda \int_a^b k(x,t)\phi(t)dt, \quad x \in [a,b] \tag{1.1}$$

where $f(t)$ and the kernels $k_1(t,s)$ and $k_2(t,s)$ are assumed to be in $L^2(R)$.

In this paper, we are introduced a numerical solution of nonlinear Fredholm integro differential equation of the form:

$$\phi'(x) = f(x) + \lambda \int_a^b F(x,t,\phi(t))dt, \qquad 0 \le x,t \le 1 \tag{1.2}$$

where $f(x)$, and $F(x,t,\phi(t))$ are assumed to be in $L^2(R)$, and satisfy the Lipschitz condition: [. El-Ameen and El-Kady, 2012]

$$\left|k(x,t,\phi_1(t)) - k(x,t,\phi_2(t))\right| \le N(x,t)\left|\phi_1(x) - \phi_2(x)\right| \tag{1.3}$$

This work is organized as follows. In section 2, we present the reformulations of equation (1.2) by Fredholm type integro differential equation, which can be convert it to optimal control problem (OCP) and by the same way we present the reformulations of Volterra tybe integro differential equation, which can be convert it to (OPC). In section 3, the existence and uniqueness are presented. The computational results are shown in section 4.

Problem Statements and Reformulation:

In this section, we introduce a numerical method for solving nonlinear Fredholm integro differential equations and Volterra integro differential equations.

Fredholm Integro Differential equation:

Let the F given in (1.2) can be rewrite in the form: [Hosseini and Shahmorad, 2009]

$$\phi'(x) = f(x) + \lambda \int_a^b k(x,t,\phi(t))\, dt \tag{2.1.1}$$

where *f(x)* is a specified function , $\phi(x)$ the functional to be solved and $k(x,t,\phi(t))$ is the integral kernel. and by considering

$$\phi'(x) = \psi(x), \qquad \phi(t) = \int_a^t \psi(s)\, ds \tag{2.1.2}$$

Then equation (2.1.1) will be redefine at equation (2.1.3)

$$\psi(x) = f(x) + \lambda \int_a^b k(x,t,\int_a^t \psi(s)ds)\, dt \tag{2.1.3}$$

where $\psi(x)$ is the functional to be solved and $k(x,t,\int_a^t \psi(s)ds)$ will be the integral kernel

and the kernel $k(x,t,\int_a^t \psi(s)ds) \in c[a,b]\times[a,b]$; which satisfy:

$$\left| k(x,t,\int_a^t \psi(s)ds) \right| \le M, \quad \left| f(t) \right| \le k \tag{2.1.4}$$

where M, K are arbitrary constants.

It easy to see that (2.1.3) can be written as follows:

$$\psi(x) - f(x) = \int_a^x (\psi^{\bullet}(t) - f^{\bullet}(t))\, dt + (\psi(a) - f(a))$$

$$,\psi(x) - f(x) = \int_a^b \mu(\psi^{\bullet}(t) - f^{\bullet}(t))\, dt + (\psi(a) - f(a)), \qquad where\ \mu = \begin{cases} 1 & a \prec t \prec x \prec b \\ 0 & t \succ x \end{cases} \tag{2.1.5}$$

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then

$$\int_a^b (\psi^{\bullet}(t) - f^{\bullet}(t))\ \mu\, dt + \int_a^b k(a,t,s)\, dt = \lambda \int_a^b k(x,t,\int_a^t \psi(s)\, ds)\, dt \tag{2.1.6}$$

since

$$\psi(a) - f(a) = \int_a^b k(a,t,\int_a^t \psi(s) ds)\, dt \tag{2.1.7}$$

therefore,

$$\int_a^b [(\psi^{\bullet}(t) - f^{\bullet}(t))\, \mu + \lambda\, k(a,t,\int_a^t \psi(s)\, ds) - \lambda\, k(x,t,\int_a^t \psi(s)\, ds)]\, dt = 0 \tag{2.1.8}$$

let

$$G(x) = \int_a^b [(\psi^{\bullet}(t) - f^{\bullet}(t))\, \mu + \lambda\, \{k(a,t,\int_a^t \psi(s)\, ds) - k(x,t,\int_a^t \psi(s)\, ds)\}]\, dt\ = 0 \tag{2.1.9}$$

that is

$$|G(x)| = 0 \tag{2.1.10}$$

By integrating (2.1.10) on [a,b], we have

$$\int_a^b |G(x)|\, dx = 0 \tag{2.1.11}$$

On the other hand, one can define the following equality:

$$F_{\circ}(x,t,\psi(t),u(t)) = (u(t) - f^{\bullet}(t))\mu + \lambda[k(a,t,\int_a^t \psi(s)\, ds) - k(x,t,\int_s^t \psi(s)\, ds)] \tag{2.1.12}$$

this will lead us to the following inequality:

$$\int_a^b |G(x)|\, dx \le \int\int_a^b |F_{\circ}(x,t,\psi(t),u(t))|\, dt\, dx \tag{2.1.13}$$

where

$$\psi^{\bullet}(t) = u(t), \quad t \in [a,b]. \tag{2.1.14}$$

with the boundary condition

$$\psi(a) = f(a) + \lambda \int_a^b k(a,t,\int_a^t \psi(s) ds)\, dt, \quad \psi(b) = f(b) + \lambda \int_a^b k(b,t,\int_a^t \psi(s) ds)\, dt \tag{2.1.15}$$

At the end, we have the following OCP that is minimizing functional

Minimize

$$I = \int_{\Omega} |F_0(x,t,\psi(t),u(t))| \tag{2.1.16}$$

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subject to

$$\psi'(t) = u(t), \qquad t \in [a,b], \tag{2.1.17}$$

and $\psi(b)$ are defined in (2.2.15), where $\Omega = [a,b] \times [a,b]$

minimize

$$I = \int_{\Omega} \left|F_0\left(x,t,\psi(t),u(t)\right)\right| \tag{2.1.20}$$

subject to

$$\psi'(t) = u(t), \qquad t \in [a,b]; \tag{2.1.21}$$

where $\Omega = [a,b] \times [a,b]$

$$\psi(a) = f(a) + \int_a^b k\left(a,t,\int_a^t \psi(t)\,dt\right)dt,\ \psi(b) = f(b) + \int_a^b k\left(b,t,\int_a^t \psi(t)\,dt)\right)dt \tag{2.1.22}$$

Volterra Integro Differential equation:

We introduce a method to solve a numerical method for solving nonlinear Volterra integro differential (VIDE) equation of the form [Shidfar *et. al.* 2010 ; Pour-Mahmoud *et. al.*, 2005]

$$\phi^m(x) = f(x) + \sum_{j=0}^{j} \lambda_j \int_a^x k_j\left(x,t,\phi(t)\right)dt, \quad x\epsilon[a,b] \tag{2.2.1}$$

In this paper, we are introduced a method to solving the numerical solution of a nonlinear (VIDE), without loss of generality, we take *m=1*, then FIDE in the form:

$$\phi'(x) = f(x) + \lambda \int_a^x V(x,t,\phi(t))dt \tag{2.2.2}$$

$$k\left(x,t,\phi(t)\right) = e\left(x,t,\phi(t)\right)V(x,t,\phi(t))$$
$$e\left(x,t,\phi(t)\right) = \begin{cases} 1 & a < t, x < b, \\ 0 & t > x, \end{cases} \tag{2.2.3}$$

the kernel satisfies codition in equation (2.1.4)
then equation (2.2.3) can be written in the form

$$\phi'(x) = f(x) + \lambda \int_a^b k(x,t,\phi(t))dt \tag{2.2.4}$$

Put

$$\phi'(x) = \psi(x)$$

and

$$\phi(x) = \int_a^x \psi(x)\ dx$$

$$\phi(t) = \int_a^t \psi(t)\ dt \tag{2.2.5}$$

Equation (2.2.5) can be replaced by:

$$\psi(x) = f(x) + \lambda \int_a^b k\left(x,t,\int_a^t \psi(t)\,dt\right)dt \tag{2.2.6}$$

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$$k(x,t,\int_a^x \psi(t)\,dt)$$

where $\psi(x)$ is the function to be solved and will be the integral kernel, equation (2.2.6) be the same equation(2.1.1) and satisfies all conditions of its.

$$\psi(x) - f(x) = \int_a^x (\psi'(t) - f'(t))dt - (\psi(a) - f(a))$$

$$\psi(x) - f(x) = \int_a^b \mu(\psi'(t) - f'(t))\,dt + (\psi(a) - f(a)), \quad where\ \mu = \begin{cases} 1 & a \prec t \prec x \prec b \\ 0 & t \succ x \end{cases} \tag{2.2.7}$$

Then

$$\int_a^b (\psi'(t) - f'(t))\mu dt - (\psi(a) - f(a)) = \lambda \int_a^b k\left(x,t,\int_a^t \psi(t)\,dt\right)dt \tag{2.2.8}$$

Science

$$\psi(a) - f(a) = \lambda \int_a^b k\left(a,t,\int_a^t \psi(t)\,dt\right)dt \tag{2.2.9}$$

Therefore

$$\int_a^b (\psi'(t) - f'(t))\mu dt - \int_a^b k\left(a,t,\int_a^t \psi(t)\,dt\right)dt = \int_a^b k\left(x,t,\int_a^t \psi(t)\,dt\right)dt \tag{2.2.10}$$

Therefore,

$$\int_a^b \left[\{\mu(\psi'(t) - f'(t))\} - k\left(a,t,\int_a^t \psi(t)\,dt\right) - k(x,t,\int_a^b \psi(t)\,dt)\right] = 0 \tag{2.2.11}$$

Let

$$G(x) = \int_a^b \left[\{\mu(\psi'(t) - f'(t)\} - k\left(a,t,\int_a^t \psi(t)\,dt\right) - k(x,t,\int_a^b \psi(t)\,dt)\right] = 0 \tag{2.2.12}$$

that is, if

$$\left|G(x)\right| = 0 \tag{2.2.13}$$

by integrating the equality in (2.2.13), we have

$$\int_a^b \left|G(x)\right| dx = 0 \tag{2.2.14}$$

on the other hand, one can define the following equality:

$$F_\circ(x,t,\psi(t),u(t)) = \left[(u(t) - f'(t))\right]\mu - k\left(a,t,\int_a^t \psi(t)\,dt\right) - k(x,t,\int_a^t \psi(t)\,dt) \tag{2.2.15}$$

this will lead us to the following inequality:

$$\int_a^b \left|G(x)\right| dx \le \int\int_a^b \left|F_\circ(x,t,\psi(t),u(t))\right| dt\,dx \tag{2.2.16}$$

Where

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$$\psi^{\square}(t) = u(t), \quad t \in [a,b] \tag{2.2.17}$$

with the boundary condition

$$\psi(a) = f(a) + \int_a^b k\left(a,t,\int_a^t \psi(t)\,dt\right)dt, \quad \psi(b) = f(b) + \int_a^b k\left(b,t,\int_a^t \psi(t)\,dt)\right)dt \tag{2.2.18}$$

at the end, we have the following OCP that is minimizing functional

minimize

$$I = \int_{\Omega} \left| F_{\circ}(x,t,\psi(t),u(t)) \right| \tag{2.2.19}$$

subject to

$$\psi^{\square}(t) = u(t), \qquad t \in [a,b]; \tag{2.2.20}$$

where

$$\Omega = [a,b] \times [a,b]$$

$\psi(a)$ *and* $\psi(b)$ *are defined at* (2.2.18) where $\Omega \in [a,b] \times [a,b]$

The existence and uniqueness of equations (2.1) will be considered in the next section by using the successive approximation method.

Existence and Uniqueness:

The solution $\int_a^t \psi(t)\,dt$ of (2.1.1) or (2.2.6) can be approximated successively as follows:

$$\psi(x) = f(x) + \lambda \int_a^b k\left(x,t,\int_a^t \psi(t)\,dt\right)dt \tag{3.1}$$

In which $f(x):[0,\alpha) \to R^n$ is continuous and $k(x,t,\int_a^t \psi(t)\,dt)$ is a $n \times n$ matrix of function continuous

Thus, we obtain sequence of functions $\psi_0(x), \psi_1(x), \cdots, \psi_n(x)$ such that

$$\psi_n(x) - \lambda \int_a^b k(x,t,\int_a^t \psi_{n-1}(t))\,dt = f(x) \tag{3.2}$$

with $\psi_0(x) = f(x)$

It is convenient to introduce

$$\varphi_n(x) = \psi_n(x) - \psi_{n-1}(x), \quad n \geq 1, \tag{3.3}$$

with $\varphi_0(x) = f(x)$

subtracting from(3.2), the same equation with replacing n by n-1, we get

$$\psi_n(x) - \psi_{n-1}(x) = \lambda \int_a^b k\left(x,t,\int_a^t \psi_{n-1}(t)\,dt\right)dt - \lambda \int_a^b k\left(x,t,\int_a^t \psi_{n-2}(t)\,dt\right)dt \tag{3.4}$$

Using (3.3), we have

$$\varphi_n(x) = \lambda \int_a^b k\left(x,t,\int_a^t \varphi_{n-1}(t)\,dt\right)dt, \qquad n \geq 1. \tag{3.5}$$

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Also; from (3.3), we deduce that $\psi_n(x) = \sum_{i=0}^{n} \varphi_i(x)$ (3.6)

The existence and uniqueness of the solution can be followed.

Theorem 3.1 *If the kernel* $k(x,t,\int_a^t \psi(t)\,dt)$ *and the function f(x) are continuous and satisfy condition (*2.1.4) *in* $a < x < t < b$, *then the integral equation (*2.1.3) *possesses a unique continuous solution.*
Proof .From (3.5), we get

$$|\varphi_n(x)| = \left|\lambda \int_a^b k\left(x,t,\int_a^t \varphi_{n-1}(t)\,dt\right)dt\right|$$

$$\leq \lambda \left|k(x,t,\int_a^t \varphi_{n-1}(t)\,dt)\right| \int_a^b dt$$

$$\leq \lambda(b-a)M$$

We now show that this $\int_s^x \psi(x)\,dx$ satisfies (2.1.3).
The series (3.6) is uniformly convergent since the term $\varphi_i(x)$ is dominated by $\lambda(b-a)M$.

Then,

$$\lambda \int_a^b k\left(x,t,\sum_{i=0}^{\infty}\int_a^t \varphi_i(t)\,dt\right)dt = \sum_{i=0}^{\infty} \lambda \int_a^b k\left(x,t,\int_a^t \varphi_i(t)\,dt\right)dt$$

$$= \sum_{i=0}^{\infty} \varphi_{i+1}(x) = \sum_{i=0}^{\infty} \varphi_{i+1}(x) + \varphi_0(x) - \varphi_0(x) \tag{3.8}$$

Hence, we have

$$\lambda \int_a^b k\left(x,t,\sum_{i=0}^{\infty}\int_a^t \varphi_i(t)\,dt\right)dt = \sum_{i=0}^{\infty} \varphi_i(x) - f(x) \tag{3.9}$$

This proves that $\int_a^x \psi(x)\,dx$ defined in (3.6), satisfies (2.1.3). Since each of the $\int_a^x \varphi_i(x)\,dx$ is clearly continuous, therefore $\int_a^x \psi(x)\,dx$ is continuous, where it is the limit of a uniformly convergent sequence of continuous functions.

To show that $\int_a^x \psi(x)\,dx$ is a unique continuous solution, suppose that there exists another continuous solution $\int_a^x \tilde{\psi}(x)\,dx$ of (2.1.3), then,

$$\tilde{\psi}(x) - \lambda \int_a^b k\left(x,t,\int_a^t \tilde{\psi}(t)\,dt\right)dt = f(x) \tag{3.10}$$

Subtracting (3.10) from (3.1), we get

$$\psi(x) - \tilde{\psi}(x) = \lambda \int_a^b k\left(x,t,\int_a^t \{\psi(x) - \tilde{\psi}(x)\}dt\right)dt. \tag{3.11}$$

Since $\int_a^x \psi(x)\,dx$ and $\int_a^x \tilde{\psi}(x)\,dx$ are both continuous, there exists a constant *B* such that

$$\left|\int_a^x \psi(x)\,dx - \int_a^b \tilde{\psi}(x)\,dx\right| \leq B. \tag{3.12}$$

By using the condition of (2.1.4), the inequality (3.12) becomes

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$$\left|\int_a^x \psi(x)\,dx - \int_a^x \tilde{\psi}(x)\,dx\right| \le \lambda(b-a)MB. \tag{3.13}$$

For the large enough *n*, the right-hand side is arbitrary small, then

$$\phi(x) = \tilde{\phi}(x). \tag{3.14}$$

This completes the proof.

Computational results:

In this section numerical experiments will be carried out in order to compare the performances of the new method with respect to the classical collocation methods. The method has been applied to the following three test problems [Golbabai and Seifollahi, 2007 ; El-Kady *et. al.*, 2009]:

***Example 4.1*:** Consider the Fredholm integro differential equation

$$y'(x) = xe^x + e^x - x + \int_0^1 x\, y(t)\,dt, \tag{4.1}$$

with the initial condition

$y(0) = 0$,

with exact solution

$y(x) = x\,e^x$.

We applied the method presented in this paper and solved Eq. (4.1). The computational results together with the exact solution $y(x)=e^{-3x}$ are given in figure (1). It presents a comparison between the absolute error obtained by ultraspherical integration method [El-Kady, 2009], the results obtained by using radial basis function (RBF) [Darania, 2007] networks and via optimal control method in figure (2).

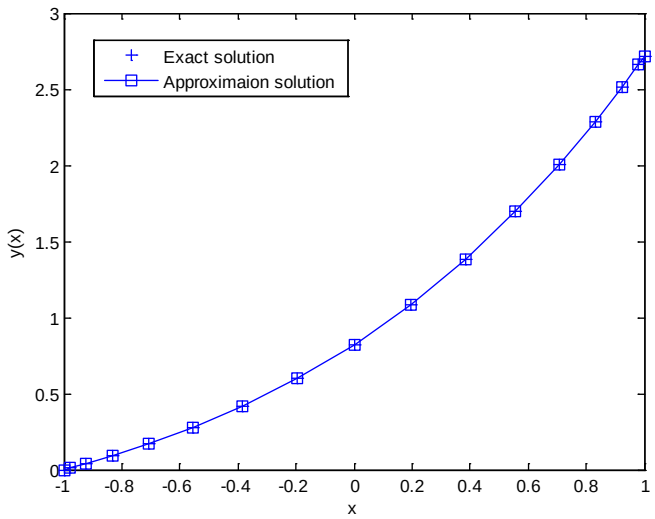

Fig. 1: Observed results for Example 4.1 (N=16).

Table 1: Convergence of OC method with different values of N.

N	Errors in OCP
6	2.444E-07
8	1.924E-10
12	6.437E-13
16	1.023E-14
20	2.927E-15

The computational maximum absolute errors for different values N are shown in Table (1). It is clear the optimal control method is more accurate for large values of N. It presents a comparison between the absolute error obtained by ultraspherical integration method, radial basis function (RBF) networks [Darania] and via

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optimal control problem at (N=10) in figure (2). The results show that the efficiency and spectral accuracy of optimal control method (OC).

***Example 4.2*:** Consider the first order Volterra integro differential equation

$$y'(x) = 1 + 2x - y(x) + \int_0^x x(1+2x)e^{t(x-t)} y(t)dt, \quad 0 \le x, t \le 1 \tag{4.2}$$

with initial condition $y(0) = 1$. The exact solution is: $y(x) = e^{x^2}$.

We solved Eq. (4.2) using the OC method presented in this paper. The computational results together with the exact solution $y(x) = e^{x^2}$ are given in figure(3), and presents a comparison between the absolute error obtained by ultraspherical integration method[El-Kady, *et. al.*, 2009], the results obtained by using radial basis function (RBF) [Darania, 2007] networks and via optimal control problem in figure(4)

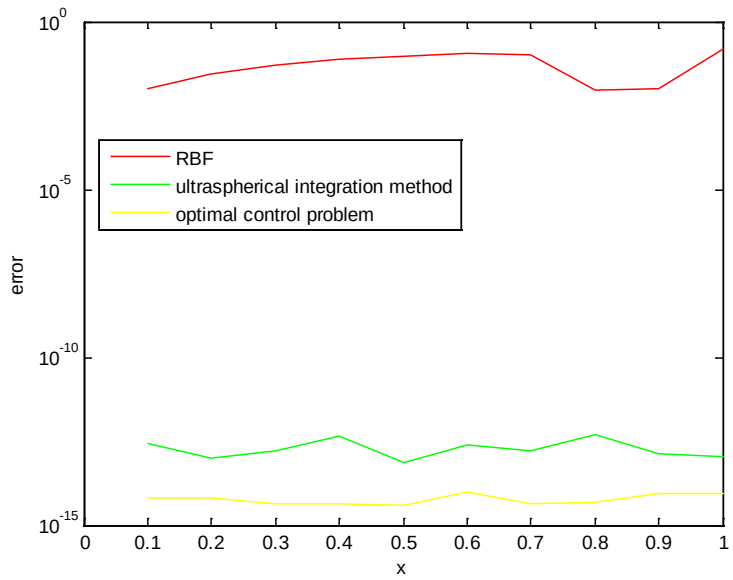

Fig. 2: Observed Compare solutions with the results Example 4.1 for different methods (N=10).

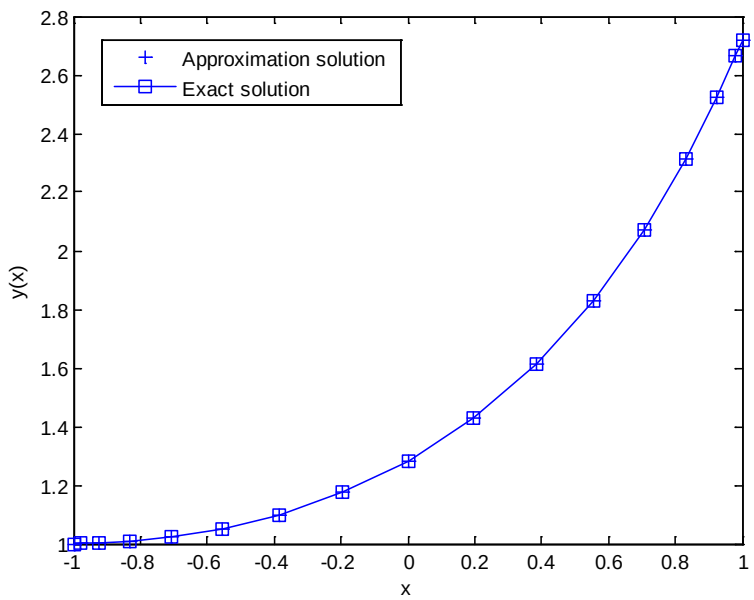

Fig. 3: Observed results for Example4.2 (N=16).

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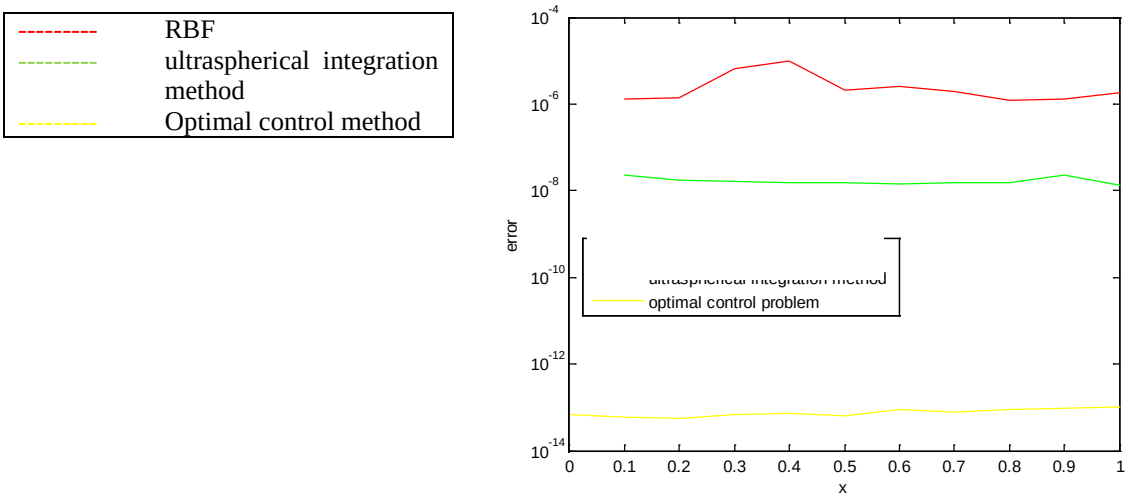

Fig. 4: Observed Compare solutions with the results Example 4.1 for different methods (N=10).

Table 2: Convergence of OC method with different values of N

N	Errors OCP
4	4.034E-03
6	3.216E-5
8	7.482E-07
12	6.066E-10
16	1.000E-13

The computational maximum absolute errors for different values N are shown in Table (2). It is clear the optimal control method is more accurate for large values of N. The numerical solutions are computed by two methods and summarized in Figure 4 for (N=10), and it seems that our method compared very well with those obtained via optimal method.

***Example 4.3*:** Consider the third order integro differential equation **[**El-Kady, *et. al.*, 2009**]**

$$y'''(s)+\int_0^{\pi/2} s\,\tau\, y'(\tau)\,d\tau=\sin s()-s, \tag{4.3}$$

with the initial conditions

$$y(0)=1\,,\quad y'(0)=0,\quad y''(0)=-1.$$

The exact solution is

$$y(s)=\cos(s)\,.$$

Transforming the domain $s,\tau\in[0,\frac{\pi}{2}]$ to $x,t\in[-1,1]$, where

$$s=\tfrac{\pi}{4}(x+1),\ \tau=\tfrac{\pi}{4}(t+1)\,.$$

Then the third order integro differential equation become

$$(\tfrac{4}{\pi})^3 y'''(x)+(\tfrac{\pi}{4})^2\int_{-1}^{1}(x+1)(t+1)\,y'(t)\,dt=\sin(\tfrac{\pi}{4}(x+1))-\tfrac{\pi}{4}(x+1),$$

with the conditions

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$$y(-1)=1\ , \qquad y'(-1)=0, \quad 16y''(-1)=-\pi^2.$$

The exact solution is

$$y(x)=\cos(\tfrac{\pi}{4}(x+1)).$$

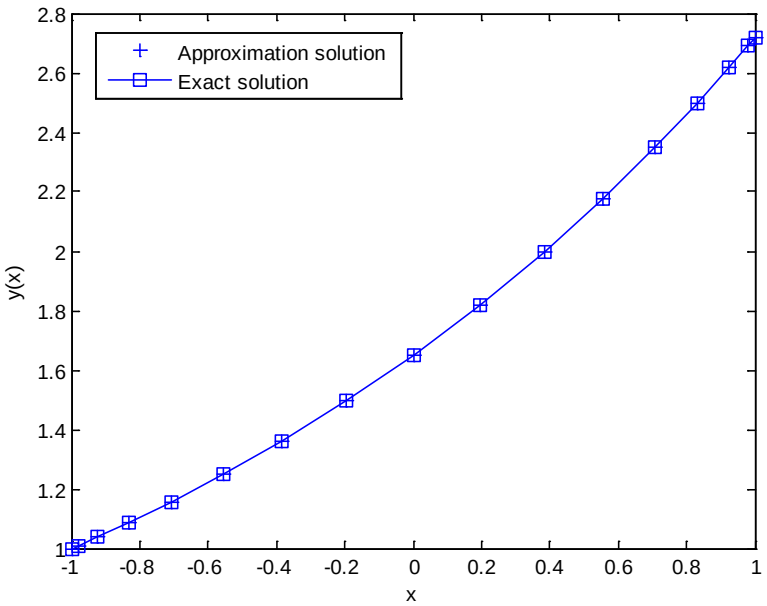

Fig. 3: Observed results for Example 4.3 (N=16).

Table 3: Convergence of OC method with different values of N

N	Errors in optimal control method
4	4.034E-03
6	3.216E-5
8	7.482E-07
12	6.066E-10
16	1.000E-13

The computational maximum absolute errors for different values N are shown in Table (3). It is clear the optimal control method is more accurate for large values of N.

Conclusion:

Some practical problems lead to linear integro-differential equations. These types of equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. As shown by numerical examples, the method introduced here can be simply implemented to general non-linear integro differential equations of the second kind. We have shown that by simple, also fast, algorithm we can gain the better results obtained by others.

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