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Calculating the relative efficiency of two-component DMUs under variable returns to scale

Ali Payan and Mahdi Shahraki Ghadimi

Department of Mathematics, Zahedan Branch, Islamic Azad University, Zahedan, Iran.

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ABSTRACT

In this study, a method is presented in order to calculate the relative efficiency of two-component decision making units (DMUs) under variable returns to scale (VRS). Numerous methods have been presented so far, but none of the guarantees obtaining the relative performance of decision-making units. Lack of calculation of relative efficiency causes basic problems in the analysis of DMUs such as lack of efficiency boundary production and we can thus mention the lack of presenting the pattern units for inefficient ones. One of the most-widely used and common presented methods to evaluate two-component DMUs will be modified so that it can calculate the performance of these units based on the relative efficiency under variable returns to scale. Calculating the relative efficiency of the components is also considered here in this study. Finally, an illustration is presented to expound and clarify the method.

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INTRODUCTION

Data envelopment analysis (DEA) is applied as a non-parametric method to estimate the efficiency of decision making units (DMU). This method was first provided by Charnes *et al*. [4] as the CCR model to calculate the efficiency of DMUs under constant returns to scale (CRS) assumption and it was rapidly expanded and used to assess several organizations and industries. Data envelopment analysis was developed by Banker *et al*. [2] under variable returns to scale (VRS) assumption. The basic argument in the data envelopment analysis is the efficiency of units which is one of the profound economic implications. Efficiency shows how an organization has made use of the institutions to produce the staff's optimum and in other words, it is indicative of "doing the work properly". According to definition by Cooper and colleagues, efficiency is calculated by the following equation:

Efficiency = Weighted sum of outputs / Weighted sum of inputs.

In the DEA if efficiency of under evaluation unit is equal to one, that unit is considered to be efficient.

Most DEA models are designed to calculate the score efficiency of single component units. In these models only an overview of performance measurement is provided, but in most practical applications, DMUs may be divided into different components called multi-component DMUs and each component uses the contribution of inputs to produce some of outputs. Some of the inputs and outputs of units are shared in two or more components. For example, universities can be considered as multi-component DMUs of which their components are the financial, research and educational activities. The number of teachers as an input is shared in research and educational components. This index is effective in producing the training outputs such as the number of graduates and also in producing research outputs such as the number of scientific papers.

Thus, it requires extending DEA models for evaluation of DMUs with these structures. For the first time, Cook *et al*. [5] used a model for measuring the efficiency of multi-component DMUs with shared inputs. They used the presented model for the major Canadian banks branches whose components include sales and service activities. Amirteimuri and Kordrostami [1] measured the efficiency of multi-component DMUs with ambiguous data as to maintain its linearity. Jahanshahloo *et al*. [7], at first, studied performance analysis of components in multi-components DMUs. Then they measured the efficiency by grouping branches of banks according to their organizational role. Jahanshahloo *et al*. [8] measured the efficiency of multi-components DMUs when there are shared inputs and outputs.

Corresponding Author: Ali Payan, Department of Mathematics, Zahedan Branch, Islamic Azad University, Zahedan, Iran.

The main problem in the presented models to calculate the efficiency of multi-component units is that it is not always possible to measure relative performance of multi-component units, and to calculate the absolute efficiency of existing models due to the their nature. In this paper, an example is provided to illustrate this claim. In this example, the achieved efficiencies of Cook *et al*.'s model [5] under VRS show that this model is not able to calculate the relative efficiency of two-component units. This issue causes the major problems in the analysis of DMUs among which may be mentioned the lack of production frontier and therefore did not provide benchmark for inefficient units and they did not measure return to scale in efficient units. Accordingly, a model to obtain the relative efficiency in two-component DEA in the case of VRS is proposed. Also according to the presented model, a method is described to calculate the relative efficiency of components based on their priority that is determined by the decision maker and thus the possibility to compare similar components in multi-component units are discussed.

The subsequent sections of this paper are as follows: The second section introduces data envelopment analysis with multi-component units. The third section consists of calculating the relative efficiency in two-component DMUs. In the fourth, method of calculating the efficiency of components based on the relative efficiency of two-component units is introduced. Then, an example is provided to illustrate the method. Finally, in the last section of this article, the results and the proposed topics for upcoming research are described.

Two-component DEA under variable returns to scale:

Consider n two-component DMUs. Suppose a shared input vector and a shared output vector are between the two components. In other words, the two components are involved in using shared inputs and shared outputs. The input vectors of the first and second components of the *kth* DMU $(k=1,\dots,n)$ are shown as follows:

$$x_k^{\ 1}=\left(x_{1k}^{\ 1},x_{2k}^{\ 1},\dots,x_{m_1k}^{\ 1}\right)^t,\ \ x_k^{\ 2}=\left(x_{1k}^{\ 2},x_{2k}^{\ 2},\dots,x_{m_2k}^{\ 2}\right)^t,$$

and their output vectors of the *kth* DMU $(k=1,\dots,n)$ are as follows:

$$y_k^{\ 1}=\left(y_{1k}^{\ 1},y_{2k}^{\ 1},\dots,y_{s_1k}^{\ 1}\right)^t,\ \ y_k^{\ 2}=\left(y_{1k}^{\ 2},y_{2k}^{\ 2},\dots,y_{s_2k}^{\ 2}\right)^t,$$

and the common input vector and the common output vector of *kth* unit $(k=1,\dots,n)$ are called $y_k^{\ c}, x_k^{\ c}$ so that:

$$x_k^{\ c}=\left(x_{1k}^{\ c},x_{2k}^{\ c},\dots,x_{mk}^{\ c}\right)^t,\ \ y_k^{\ c}=\left(y_{1k}^{\ c},y_{2k}^{\ c},\dots,y_{sk}^{\ c}\right)^t,$$

The global outline of the production process for a two-component DMU is shown in the following figure as:

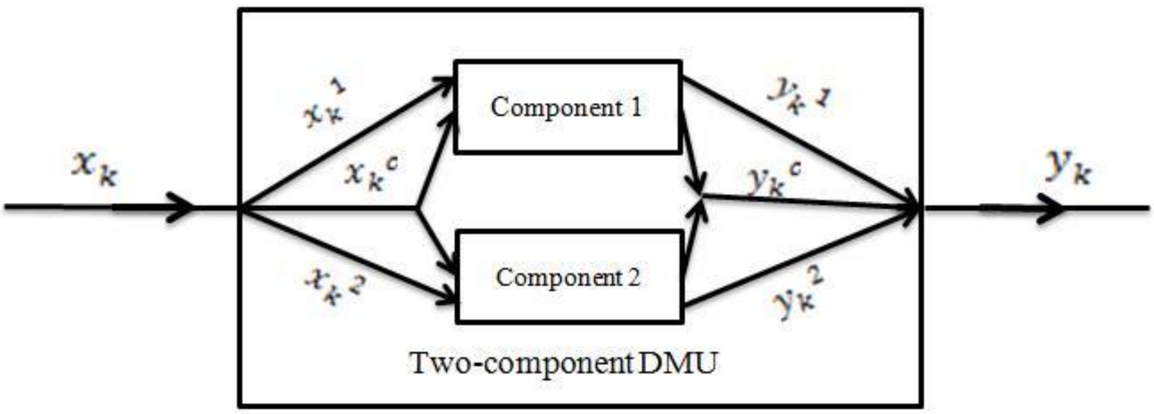

Fig. 1: Schematic diagram of the production process for the two-component unit

To obtain the overall performance of a two-component DMU based on the performance of its components, at first, the influence of the shared inputs and shared outputs on each component must be specified. Suppose that the assigned portion of the shared input to the first component are to be shown by a factor $0\le\alpha\le1$ and its assigned portion to the second component is then determined by a factor $1-\alpha$ Therefore, the shared input vector $x_k{}^c$ is broken into two vectors αx_k^c and $(1-\alpha)x_k^c$ the first and second component, respectively. In this respect we have:

$$\alpha x_k^c = \left(\alpha x_{1k}{}^c, \alpha x_{2k}{}^c, ..., \alpha x_{mk}{}^c\right)^t, (1-\alpha)x_k^c = \left((1-\alpha)x_{1k}{}^c, (1-\alpha)x_{2k}{}^c, ..., (1-\alpha)x_{mk}{}^c\right)^t.$$

Similarly, if the assigned portion of the shared output to the first component is shown by a factor $0 \le \beta \le 1$ and the portion of that to the second component is then determined by a factor $(1-\beta)$, therefore, shared output vector $y_k{}^c$ is broken into the two vectors βy_k^c and $(1-\beta)y_k^c$ for the first and second components where:

$$\beta y_k^c = \left(\beta y_{1k}{}^c, \beta y_{2k}{}^c, ..., \beta y_{sk}{}^c\right)^t, (1-\beta)y_k^c = \left((1-\beta)y_{1k}{}^c, (1-\beta)y_{2k}{}^c, ..., (1-\beta)y_{sk}{}^c\right)^t.$$

Definition 1:

The overall efficiency and component efficiency of two-component units in the VRS are defined as follows:

$$e_k^a = \frac{\left(\mu^1 y_k^1 + \mu^{c1}\beta y_k^c\right) + \left(\mu^2 y_k^2 + \mu^{c2}(1-\beta)y_k^c\right) + \mu_0}{\left(v^1 x_k^1 + v^{c1}\alpha x_k^c\right) + \left(v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c\right)}, \quad k = 1,...,n,$$

$$e_k^1 = \frac{\mu^1 y_k^1 + \mu^{c1}\beta y_k^c + \mu_1}{v^1 x_k^1 + v^{c1}\alpha x_k^c}, \quad k = 1,...,n,$$

$$e_k^2 = \frac{\mu^2 y_k^2 + \mu^{c2}(1-\beta)y_k^c + \mu_2}{v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c}, \quad k = 1,...,n,$$

where e_k^a ، e_k^1 and e_k^2 are the overall efficiency and they are the efficiency of the first and second components, respectively.

Theorem1:

The convex combination of the first and second component efficiencies when $\mu_1 + \mu_2 = \mu_0$ is the overall efficiency of two-component DMU.

Proof. Considering $\gamma_k = \dfrac{v^1 x_k^1 + v^{c1}\alpha x_k^c}{v^1 x_k^1 + v^{c1}\alpha x_k^c + v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c}$, we have

$$\gamma_k e_k^1 + (1-\gamma_k)e_k^2$$

$$= \frac{v^1 x_k^1 + v^{c1}\alpha x_k^c}{v^1 x_k^1 + v^{c1}\alpha x_k^c + v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c} \times \frac{\mu^1 y_k^1 + \mu^{c1}\beta y_k^c + \mu_1}{v^1 x_k^1 + v^{C1}\alpha x_k^C} + \left(1 - \frac{v^1 x_k^1 + v^{c1}\alpha x_k^c}{v^1 x_k^1 + v^{c1}\alpha x_k^c + v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c}\right) \times \frac{\mu^2 y_k^2 + \mu^{c2}(1-\beta)y_k^c + \mu_2}{v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c}$$

$$= \frac{\mu^1 y_k^1 + \mu^{c1}\beta y_k^c + \mu_1}{v^1 x_k^1 + v^{c1}\alpha x_k^c + v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c} + \frac{\mu^2 y_k^2 + \mu^{c2}(1-\beta)y_k^c + \mu_2}{v^1 x_k^1 + v^{c1}\alpha x_k^c + v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c}$$

$$= \frac{\mu^1 y_k^1 + \mu^{c1}\beta y_k^c + \mu^2 y_k^2 + \mu^{c2}(1-\beta)y_k^c + \mu_1 + \mu_2}{v^1 x_k^1 + v^{c1}\alpha x_k^c + v^2 x_k^2 + v^{c2}(1-\alpha)x_k^c} = e_k^a.$$

The following model is an extension of the model of Cook *et al*. [5] to calculate the efficiency of two-component DMUs under VRS assumption based on Definition 1.

$$\max \ e_o{}^a,$$

$$s.t \ e_k{}^a \le 1, \ k = 1,...,n,$$

$$e_k{}^1 \le 1, \ k = 1,...,n,$$

$$e_k{}^2 \le 1, \ k = 1,...,n,$$

$$\mu_0 = \mu_1 + \mu_2,$$

$$0 \le \alpha, \beta \le 1,$$

$$\mu^1 \geq 0, \mu^2 \geq 0, \mu^{c1} \geq 0, \mu^{c2} \geq 0, v^1 \geq 0, v^2 \geq 0, v^{c1} \geq 0, v^{c2} \geq 0, \mu_0, \mu_1, \mu_2 \, free \tag{1}$$

In the above model, this issue is noting that based on two constraints $e_k{}^2 \geq 1,\ e_k{}^1 \geq 1\ (k = 1,\ldots,n)$ constraint $e_k{}^a \geq 1\ (k = 1,\ldots,n)$ is redundant and therefore it can be eliminated from the model.

Relative efficiency of two-component DMUs under variable returns to scale:

We consider the data in Table 1 and calculate the efficiency of two-component DMUs by using model (1). The performance of the two-component units which have a shared input and two independent outputs are reported in the fifth column of the below Table 1.

Table 1: Data of two-component DMUs

DMU	x^c	y_1	y_2	Efficiency of model (1)	Efficiency of new model (9)
A	400	8000	500	0.905	1.000
B	400	400	10000	0.924	1.000

As it can be seen above, the efficiency of both units is less than one. Lack of an efficient unit (a unit with efficiency equal to one) represents the inability of the model to calculate the relative efficiency of two-component units. Based on the nature of DEA, efficient frontier is constructed by efficient units. Thus, in this state, efficient frontier cannot be found. On the other hands, the obtained performances does not seem to be realistic. Two units use the same input value and unit A in component one produces an output which is twentyfold of that of component two and in this direction the performance of A is twentyfold of that of B. A similar analysis for the second components of two units depicts that the performance of B is twentyfold of A. Thus, it is not logical to say that the performance of B is better than A, while the results in the fifth column of Table 1 upholds the higher performance of A rather than B. To solve the mentioned problems we take the following measures.

The following model to calculate the relative efficiency of two-component DMUs under VRS is proposed as:

$$\begin{aligned}
& max \ \frac{e_o^a}{\max\limits_{k=1,\ldots,n} \left\{ e_k^a \right\}}, \\
& s.t \ \ e_k{}^1 \leq 1, \ k = 1,\ldots,n, \\
& e_k{}^2 \leq 1, \ k = 1,\ldots,n, \\
& \mu_0 = \mu_1 + \mu_2 \ , \\
& 0 \leq \alpha, \beta \leq 1, \\
& \mu^1 \geq 0, \mu^2 \geq 0, \mu^{c1} \geq 0, \mu^{c2} \geq 0, v^1 \geq 0, v^2 \geq 0, v^{c1} \geq 0, v^{c2} \geq 0, \mu_0, \mu_1, \mu_2 \, free
\end{aligned} \tag{2}$$

By the replacement of definition 1, the following model is obtained as:

$$\begin{aligned}
& max \ \frac{\dfrac{\mu^1 y_o^1 + \mu^{c1}\beta y_o^c + \mu^2 y_o^2 + \mu^{c2}(1-\beta) y_o^c + \mu_1 + \mu_2}{v^1 x_o^1 + v^{c1}\alpha x_o^c + v^2 x_o^2 + v^{c2}(1-\alpha) x_o^c}}{\max\limits_{k=1,\ldots,n} \left\{ \dfrac{\mu^1 y_k^1 + \mu^{c1}\beta y_k^c + \mu^2 y_k^2 + \mu^{c2}(1-\beta) y_k^c + \mu_1 + \mu_2}{v^1 x_k^1 + v^{c1}\alpha x_k^c + v^2 x_k^2 + v^{c2}(1-\alpha) x_k^c} \right\}}, \\
& s.t \ \ \frac{\mu^1 y_k^1 + \mu^{c1}\beta y_k^c + \mu_1}{v^1 x_k^1 + v^{c1}\alpha x_k^c} \leq 1, \ k = 1,\ldots,n, \\
& \frac{\mu^2 y_k^2 + \mu^{c2}(1-\beta) y_k^c + \mu_2}{v^2 x_k^2 + v^{c2}(1-\alpha) x_k^c} \leq 1, \ k = 1,\ldots,n, \\
& 0 \leq \alpha, \beta \leq 1, \\
& \mu^1 \geq 0, \mu^2 \geq 0, \mu^{c1} \geq 0, \mu^{c2} \geq 0, v^1 \geq 0, v^2 \geq 0, v^{c1} \geq 0, v^{c2} \geq 0, \mu_1, \mu_2 \, free
\end{aligned} \tag{3}$$

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It should be noted that the above model is a fractional model by the purpose of maximizing the relative efficiency so that the absolute efficiency of any component does not exceeded one. By the definition of variable

$$t=\max_{k=1,\dots,n}\left\{\frac{\mu^1 y_k^1+\mu^{c1}\beta y_k^c+\mu^2 y_k^2+\mu^{c2}\left(1-\beta\right)y_k^c+\mu_1+\mu_2}{v^1 x_k^1+v^{c1}\alpha x_k^c+v^2 x_k^2+v^{c2}\left(1-\alpha\right)x_k^c}\right\},$$

model (3) will be modified to the following model:

$$max\ \frac{\dfrac{\mu^1 y_o^1+\mu^{c1}\beta y_o^c+\mu^2 y_o^2+\mu^{c2}\left(1-\beta\right)y_o^c+\mu_1+\mu_2}{v^1 x_o^1+v^{c1}\alpha x_o^c+v^2 x_o^2+v^{c2}\left(1-\alpha\right)x_o^c}}{t},$$

$$s.t\ \ t=\max_{k=1,\dots,n}\left\{\frac{\mu^1 y_k^1+\mu^{c1}\beta y_k^c+\mu^2 y_k^2+\mu^{c2}\left(1-\beta\right)y_k^c+\mu_1+\mu_2}{v^1 x_k^1+v^{c1}\alpha x_k^c+v^2 x_k^2+v^{c2}\left(1-\alpha\right)x_k^c}\right\},$$

$$\frac{t\mu^1 y_k^1+t\mu^{c1}\beta y_k^c+t\mu_1}{v^1 x_k^1+v^{c1}\alpha x_k^c}\le t,\quad k=1,\dots,n,$$

$$\frac{t\mu^2 y_k^2+t\mu^{c2}\left(1-\beta\right)y_k^c+t\mu_2}{v^2 x_o^2+v^{c2}\left(1-\alpha\right)x_o^c}\le t,\quad k=1,\dots,n,$$

$$0\le\alpha,\beta\le1,$$

$$\mu^1\ge0,\mu^2\ge0,\mu^{c1}\ge0,\mu^{c2}\ge0,v^1\ge0,v^2\ge0,v^{c1}\ge0,v^{c1}\ge0,t\ge0,\mu_1,\mu_2\,free \tag{4}$$

With the change of variables

$$\mu'^1=t\mu^1,\mu'^2=t\mu^2,\mu'^{c1}=t\mu^{c1},\mu'^{c2}=t\mu^{c2},v'^1=tv^1,v'^2=tv^2,v'^{c1}=tv^{c1},v'^{c2}=tv^{c2},\mu'_1=t\mu_1,\mu'_2=t\mu_2,$$

model (4) is converted into the following fractional model:

$$max\ \frac{\mu^1 y_o^1+\mu^{c1}\beta y_o^c+\mu^2 y_o^2+\mu^{c2}\left(1-\beta\right)y_o^c+\mu_1+\mu_2}{v'^1 x_o^1+v'^{c1}\alpha x_o^c+v'^2 x_o^2+v'^{c2}\left(1-\alpha\right)x_o^c},$$

$$s.t\ \ \frac{\mu^1 y_k^1+\mu^{c1}\beta y_k^c+\mu^2 y_k^2+\mu^{c2}\left(1-\beta\right)y_k^c+\mu_1+\mu_2}{v'^1 x_k^1+v'^{c1}\alpha x_k^c+v'^2 x_k^2+v'^{c2}\left(1-\alpha\right)x_k^c}\le1,\quad k=1,\dots,n, \tag{1-5}$$

$$\frac{\mu'^1 y_k^1+\mu'^{c1}\beta y_k^c+\mu'_1}{v'^1 x_k^1+v'^{c1}\alpha x_k^c}\le1,\quad k=1,\dots,n, \tag{2-5}$$

$$\frac{\mu'^2 y_k^1+\mu'^{c2}\left(1-\beta\right)y_k^c+\mu'_2}{v'^2 x_k^2+v'^{c2}\left(1-\alpha\right)x_k^c}\le1,\quad k=1,\dots,n, \tag{3-5}$$

$$0\le\alpha,\beta\le1,$$

$$\mu^1\ge0,\mu^2\ge0,\mu^{c1}\ge0,\mu^{c2}\ge0,\mu'^1\ge0,\mu'^2\ge0,\mu'^{c1}\ge0,\mu'^{c2}\ge0,v'^1\ge0,v'^2\ge0,v'^{c1}\ge0,v'^{c2}\ge0,$$

$$\mu_1,\mu_2,\mu'_1,\mu'_2\ free \tag{5}$$

With the definition $\dfrac{1}{t}=v'^1 x_o^1+v'^{c1}\alpha x_o^C+v'^2 x_o^2+v'^{c2}\left(1-\alpha\right)x_o^c$ the following model is obtained as:

$$max\ \ t\mu^1 y_o^1+t\mu^{c1}\beta y_o^c+t\mu^2 y_o^2+t\mu^{c2}\left(1-\beta\right)y_o^c+t\mu_1+t\mu_2,$$

$$s.t\ \ tv'^1 x_o^1+tv'^{c1}\alpha x_o^c+tv'^2 x_o^2+tv'^{c2}\left(1-\alpha\right)x_o^c=1,$$

$$t\mu^1 y_k^1+t\mu^{c1}\beta y_k^c+t\mu^2 y_k^2+t\mu^{c2}\left(1-\beta\right)y_k^c+t\mu_1+t\mu_2-\left(tv'^1 x_k^1+tv'^{c1}\alpha x_k^c+tv'^2 x_k^2+tv'^{c2}\left(1-\alpha\right)x_k^c\right)\le0,\ k=1,\dots,n,$$

$$t\mu'^1 y_k^1+t\mu'^{c1}\beta y_k^c+t\mu'_1-\left(tv'^1 x_k^1+tv'^{c1}\alpha x_k^c\right)\le0,\ k=1,\dots,n,$$

$$t\mu'^2 y_k^2+t\mu'^{c2}\left(1-\beta\right)y_k^c+t\mu'_2-\left(tv'^2 x_k^2+tv'^{c2}\left(1-\alpha\right)x_k^c\right)\le0,\ k=1,\dots,n,$$

$$0 \le \alpha, \beta \le 1,$$
$$\mu^1 \ge 0, \mu^2 \ge 0, \mu^{c1} \ge 0, \mu^{c2} \ge 0, \mu'^1 \ge 0, \mu'^2 \ge 0, \mu'^{c1} \ge 0, \mu'^{c2} \ge 0, v'^1 \ge 0, v'^2 \ge 0, v'^{c1} \ge 0, v'^{c2} \ge 0, t \ge 0,$$
$$\mu_1, \mu_2, \mu'_1, \mu'_2 \; free \qquad (6)$$

By the following variable transformations in model (6),

$$\mu'^1 = t\mu^1, \mu'^2 = t\mu^2, \mu'^{c1} = t\mu^{c1}, \mu'^{c2} = t\mu^{c2}, \mu'_1 = t\mu_1, \mu'_2 = t\mu_2,$$
$$\tilde{\mu}^1 = t\mu'^1, \tilde{\mu}^2 = t\mu'^2, \tilde{\mu}^{c1} = t\mu'^{c1}, \tilde{\mu}^{c2} = t\mu'^{c2}, \tilde{v}^1 = tv'^1, \tilde{v}^2 = tv'^2, \tilde{v}^{c1} = tv'^{c1}, \tilde{v}^{c2} = tv'^{c2}, \tilde{\mu}_1 = t\mu'_1, \tilde{\mu}_2 = t\mu'_2,$$

we will achieve a mathematical programming as:

$$max \;\; \mu'^1 y_o^1 + \mu'^{c1} \beta y_o^c + \mu'^2 y_o^2 + \mu'^{c2} (1-\beta) y_o^c + \mu'_1 + \mu'_2,$$
$$s.t \;\; \tilde{v}^1 x_o^1 + \tilde{v}^{c1} \alpha x_o^c + \tilde{v}^2 x_o^2 + \tilde{v}^{c2} (1-\alpha) x_o^c = 1, \qquad (1\text{-}7)$$
$$\mu'^1 y_k^1 + \mu'^{c1} \beta y_k^c + \mu'^2 y_k^2 + \mu'^{c2} (1-\beta) y_k^c + \mu'_1 + \mu'_2 - \left(\tilde{v}^1 x_k^1 + \tilde{v}^{c1} \alpha x_k^c + \tilde{v}^2 x_k^2 + \tilde{v}^{c2} (1-\alpha) x_k^c\right) \le 0, \;\; k = 1, \ldots, n, \qquad (2\text{-}7)$$
$$\tilde{\mu}^1 y_k^1 + \tilde{\mu}^{c1} \beta y_k^c + \tilde{\mu}_1 - \left(\tilde{v}^1 x_k^1 + \tilde{v}^{c1} \alpha x_k^c\right) \le 0, \;\;\; k = 1, \ldots, n, \qquad (3\text{-}7)$$
$$\tilde{\mu}^2 y_k^2 + \tilde{\mu}^{c2} (1-\beta) y_k^c + \tilde{\mu}_2 - \left(\tilde{v}^2 x_k^2 + \tilde{v}^{c2} (1-\alpha) x_k^c\right) \le 0, \;\; k = 1, \ldots, n, \qquad (4\text{-}7)$$
$$0 \le \alpha, \beta \le 1,$$
$$\mu'^1 \ge 0, \mu'^2 \ge 0, \mu'^{c1} \ge 0, \mu'^{c2} \ge 0, \tilde{\mu}^1 \ge 0, \tilde{\mu}^2 \ge 0, \tilde{\mu}^{c1} \ge 0, \tilde{\mu}^{c2} \ge 0, \tilde{v}^1 \ge 0, \tilde{v}^2 \ge 0, \tilde{v}^{c1} \ge 0, \tilde{v}^{c2} \ge 0,$$
$$\tilde{\mu}_1, \tilde{\mu}_2, \mu'_1, \mu'_2 \; free \qquad (7)$$

Consider changing the following variables as:

$$\tilde{\mu}^{c1} = \beta\mu'^{c1}, \tilde{\mu}^{c2} = (1-\beta)\mu'^{c2}, \overline{\overline{v}}^{c1} = \alpha\tilde{v}^{c1}, \overline{\overline{v}}^{c2} = (1-\alpha)\tilde{v}^{c2}, \overline{\overline{\mu}}^{c1} = \beta\tilde{\mu}^{c1}, \overline{\overline{\mu}}^{c2} = (1-\beta)\tilde{\mu}^{c2}.$$

So we will get to the equivalent linear programming model as:

$$max \;\;\; \mu'^1 y_o^1 + \tilde{\mu}^{c1} y_o^c + \mu'^2 y_o^2 + \tilde{\mu}^{c2} y_o^c + \mu'_1 + \mu'_2,$$
$$s.t \;\; \tilde{v}^1 x_o^1 + \overline{\overline{v}}^{c1} x_o^c + \tilde{v}^2 x_o^2 + \overline{\overline{v}}^{c2} x_o^c = 1,$$
$$\mu'^1 y_k^1 + \tilde{\mu}^{c1} y_k^c + \mu'^2 y_k^2 + \tilde{\mu}^{c2} y_k^c + \mu'_1 + \mu'_2 - \left(\tilde{v}^1 x_k^1 + \overline{\overline{v}}^{c1} x_k^c + \tilde{v}^2 x_k^2 + \overline{\overline{v}}^{c2} x_k^c\right) \le 0, \;\; k = 1, \ldots, n,$$
$$\tilde{\mu}^1 y_k^1 + \overline{\overline{\mu}}^{c1} y_k^c + \tilde{\mu}_1 - \left(\tilde{v}^1 x_k^1 + \overline{\overline{v}}^{c1} x_k^c\right) \le 0, \;\;\; k = 1, \ldots, n,$$
$$\tilde{\mu}^2 y_k^2 + \overline{\overline{\mu}}^{c2} y_k^c + \tilde{\mu}_2 - \left(\tilde{v}^2 x_k^2 + \overline{\overline{v}}^{c2} x_k^c\right) \le 0, \;\; k = 1, \ldots, n,$$
$$\mu'^1 \ge 0, \mu'^2 \ge 0, \tilde{\mu}^{c1} \ge 0, \tilde{\mu}^{c2} \ge 0, \tilde{\mu}^1 \ge 0, \tilde{\mu}^2 \ge 0, \overline{\overline{\mu}}^{c1} \ge 0, \overline{\overline{\mu}}^{c2} \ge 0, \tilde{v}^1 \ge 0, \tilde{v}^2 \ge 0, \overline{\overline{v}}^{c1} \ge 0, \overline{\overline{v}}^{c2} \ge 0,$$
$$\tilde{\mu}_1, \tilde{\mu}_2, \mu'_1, \mu'_2 free \qquad (8)$$

With variable transformations

$$\mu'^{c1} = \tilde{\mu}^{c1}, \mu'^{c2} = \tilde{\mu}^{c2}, \tilde{\mu}^{c1} = \overline{\overline{\mu}}^{c1}, \tilde{\mu}^{c2} = \overline{\overline{\mu}}^{c2}, \tilde{v}^{c1} = \overline{\overline{v}}^{c1}, \tilde{v}^{c2} = \overline{\overline{v}}^{c2}$$

a linear model is obtained by the following simple form:

$$max \;\; \mu'^1 y_o^1 + \mu'^{c1} y_o^c + \mu'^2 y_o^2 + \mu'^{c2} y_o^c + \mu'_1 + \mu'_2,$$
$$s.t \;\; \tilde{v}^1 x_o^1 + \tilde{v}^{c1} x_o^c + \tilde{v}^2 x_o^2 + \tilde{v}^{c2} x_o^c = 1,$$
$$\mu'^1 y_k^1 + \mu'^{c1} y_k^c + \mu'^2 y_k^2 + \mu'^{c2} y_k^c + \mu'_1 + \mu'_2 - \tilde{v}^1 x_k^1 - \tilde{v}^{c1} x_k^c - \tilde{v}^2 x_k^2 - \tilde{v}^{c2} x_k^c \le 0, \;\; k = 1, \ldots, n,$$
$$\tilde{\mu}^1 y_k^1 + \tilde{\mu}^{c1} y_k^c + \tilde{\mu}_1 - \tilde{v}^1 x_k^1 - \tilde{v}^{c1} x_k^c \le 0, \;\; k = 1, \ldots, n,$$
$$\tilde{\mu}^2 y_k^2 + \tilde{\mu}^{c2} y_k^c + \tilde{\mu}_2 - \tilde{v}^2 x_k^2 - \tilde{v}^{c2} x_k^c \le 0, \;\; k = 1, \ldots, n,$$
$$\mu'^1 \ge 0, \mu'^2 \ge 0, \mu'^{c1} \ge 0, \mu'^{c2} \ge 0, \tilde{\mu}^1 \ge 0, \tilde{\mu}^2 \ge 0, \tilde{\mu}^{c1} \ge 0, \tilde{\mu}^{c2} \ge 0, \tilde{v}^1 \ge 0, \tilde{v}^2 \ge 0, \tilde{v}^{c1} \ge 0, \tilde{v}^{c2} \ge 0,$$
$$\tilde{\mu}_1, \tilde{\mu}_2, \mu'_1, \mu'_2 free \qquad (9)$$

Theorem 2:

At least one of the constraints (1-5) is active in the optimality of model (5).

Proof. Suppose $\mu'^{i*}, \mu'^{ci*}, v'^{i*}, v'^{ci*}, \mu^{i*}, \mu^{ci*}, \mu_i^*, \mu_i'^*, \beta^*, \alpha^* \ (i=1,2)$ are optimal multipliers of problem (5). Also assume that all constraints (1-5) are strict inequalities. In this case,

$$\frac{\mu^{1*}y_k^1+\mu^{c1*}\beta^*y_k^c+\mu^{2*}y_k^2+\mu^{c2*}\left(1-\beta^*\right)y_k^c+\mu^*_1+\mu^*_2}{v'^{1*}x_k^1+v'^{c1*}\alpha^*x_k^c+v'^{2*}x_k^2+v'^{c2*}\left(1-\alpha^*\right)x_k^c}<1,\ k=1,\dots,n$$

Therefore there are slack variables $\Delta_k>0 \ (k=1,...,n)$ that

$$\frac{\mu^{1*}y_k^1+\mu^{c1*}\beta^*y_k^c+\mu^{2*}y_k^2+\mu^{c2*}\left(1-\beta^*\right)y_k^c+\mu^*_1+\mu^*_2}{v'^{1*}x_k^1+v'^{c1*}\alpha^*x_k^c+v'^{2*}x_k^2+v'^{c2*}\left(1-\alpha^*\right)x_k^c}+\frac{\Delta_k(1_1y_k^1+1_2y_k^2)}{v'^{1*}x_k^1+v'^{c1*}\alpha^*x_k^c+v'^{2*}x_k^2+v'^{c2*}\left(1-\alpha^*\right)x_k^c}=1,\quad k=1,\dots,n,$$

where 1_1 and 1_2 are vectors whose components are one and the number of their elements is equal to the number of components y_k^1 and $y_k^2 \ (k=1,...,n)$, respectively. We define $\Delta=min_{k=1,\dots,n}\{\Delta_k\}$. So $\Delta>0$ and $\Delta\le\Delta_k \ (k=1,...,n)$. Thus, the above equation will be as follows:

$$\frac{\mu^{1*}y_k^1+\mu^{c1*}\beta^*y_k^c+\mu^{2*}y_k^2+\mu^{c2*}\left(1-\beta^*\right)y_k^c+\mu^*_1+\mu^*_2+\Delta(1_1y_k^1+1_2y_k^2)}{v'^{1*}x_k^1+v'^{c1*}\alpha^*x_k^c+v'^{2*}x_k^2+v'^{c2*}\left(1-\alpha^*\right)x_k^c}\le 1\ ,k=1,\dots,n.$$

Therefore:

$$\frac{\left(\mu^{1*}+\Delta 1_1\right)y_k^1+\mu^{c1*}\beta^*y_k^c+\left(\mu^{2*}+\Delta 1_2\right)y_k^2+\mu^{c2*}\left(1-\beta^*\right)y_k^c+\mu^*_1+\mu^*_2}{v'^{1*}x_k^1+v'^{c1*}\alpha^*x_k^c+v'^{2*}x_k^2+v'^{c2*}\left(1-\alpha^*\right)x_k^c}\le 1,\ k=1,\dots,n.$$

As a result,

$$\frac{\mu^{1*}y_k^1+\mu^{c1*}\beta^*y_k^c+\mu^{2*}y_k^2+\mu^{c2*}\left(1-\beta^*\right)y_k^c+\mu^*_1+\mu^*_2}{v'^{1*}x_k^1+v'^{c1*}\alpha^*x_k^c+v'^{2*}x_k^2+v'^{c2*}\left(1-\alpha^*\right)x_k^c}$$
$$<\frac{\left(\mu^{1*}+\Delta 1_1\right)y_k^1+\mu^{c1*}\beta^*y_k^c+\left(\mu^{2*}+\Delta 1_2\right)y_k^2+\mu^{c2*}\left(1-\beta^*\right)y_k^c+\mu^*_1+\mu^*_2}{v'^{1*}x_k^1+v'^{c1*}\alpha^*x_k^c+v'^{2*}x_k^2+v'^{c2*}\left(1-\alpha^*\right)x_k^c},k=1,\dots,n.$$

So the above expression for $k=o$ is also true. On the other hand, $\mu'^{i*}, \mu'^{ci*}, v'^{i*}, v'^{ci*}, \mu^{i*}+\Delta 1_i, \mu^{ci*}, \mu_i^*, \mu_i'^*,, \beta^*, \alpha^* \ (i=1,2)$ construct a feasible solution of model (5). Furthermore, the objective function value for this solution is larger than the optimal solution of problem (5) which is a contradiction. Contradiction is created from the fact that all constraints (1-5) are strictly unequal. So, we can assume that contra-positive assumption is invalid and the proof holds true.

Theorem 3:

Model (5) calculates the relative efficiency of two-component DMUs.

Proof. If $\mu'^{i*}, \mu'^{ci*}, v'^{i*}, v'^{ci*}, \mu^{i*}, \mu^{ci*}, \mu^*_i, \mu_i'^*, \beta^*, \alpha^* \ (i=1,2)$ are optimal multipliers of problem (5), then according to the previous theorem

$$\max_{k=1,\dots,n}\left\{\frac{\mu^{1*}y_k^1+\mu^{c1*}\beta^*y_k^c+\mu^{2*}y_k^2+\mu^{c2*}\left(1-\beta^*\right)y_k^c+\mu^*_1+\mu^*_2}{v'^{1*}x_k^1+v'^{c1*}\alpha^*x_k^c+v'^{2*}x_k^2+v'^{c2*}\left(1-\alpha^*\right)x_k^c}\right\}=1$$

$$\Rightarrow \frac{\mu^{1*}y_o^1+\mu^{c1*}\beta^*y_o^c+\mu^{2*}y_o^2+\mu^{c2*}\left(1-\beta^*\right)y_o^c+\mu^*_1+\mu^*_2}{v'^{1*}x_o^1+v'^{c1*}\alpha^*x_o^c+v'^{2*}x_o^2+v'^{c2*}\left(1-\alpha^*\right)x_o^c}=\frac{\dfrac{\mu^{1*}y_o^1+\mu^{c1*}\beta^*y_o^c+\mu^{2*}y_o^2+\mu^{c2*}\left(1-\beta^*\right)y_o^c+\mu^*_1+\mu^*_2}{v'^{1*}x_o^1+v'^{c1*}\alpha^*x_o^c+v'^{2*}x_o^2+v'^{c2*}\left(1-\alpha^*\right)x_o^c}}{1}$$

$$= \frac{\dfrac{\mu^{1*} y_o^1 + \mu^{c1*}\beta^* y_o^c + \mu^{2*} y_o^2 + \mu^{c2*}\left(1-\beta^*\right) y_o^c + \mu_1^* + \mu_2^*}{v'^{1*} x_o^1 + v'^{c1*}\alpha^* x_o^c + v'^{2*} x_o^2 + v'^{c2*}\left(1-\alpha^*\right) x_o^c}}{\max\limits_{k=1,\ldots,n}\left\{\dfrac{\mu^{1*} y_k^1 + \mu^{c1*}\beta^* y_k^c + \mu^{2*} y_k^2 + \mu^{c2*}\left(1-\beta^*\right) y_k^c + \mu_1^* + \mu_2^*}{v'^{1*} x_k^1 + v'^{c1*}\alpha^* x_k^c + v'^{2*} x_k^2 + v'^{c2*}\left(1-\alpha^*\right) x_k^c}\right\}}$$

So the optimal value of the objective function of problem (5) is equal to the relative efficiency of DMU_o .

Theorem 4:
At least one of the constraints (2-7) is active in the optimality of model (7).

Proof. Suppose $\mu'^{i*}, \mu'^{ci*}, \tilde{v}^{i*}, \tilde{v}^{ci*}, \tilde{\mu}^{i*}, \tilde{\mu}^{ci*}, \tilde{\mu}_i^*, \mu_i'^*, \alpha^*, \beta^* \quad (i=1,2)$ are optimal multipliers of problem (7). Also assume that all constraints (2-7) are as strict inequalities. In this case,

$$\mu'^{1*} y_k^1 + \mu'^{c1*}\beta^* y_k^c + \mu'^{2*} y_k^2 + \mu'^{c2*}\left(1-\beta^*\right) y_k^c + \mu_1'^* + \mu_2'^* - \left(\tilde{v}^{1*} x_k^1 + \tilde{v}^{c1*}\alpha^* x_k^c + \tilde{v}^{2*} x_k^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_k^c\right) < 0,\ k=1,\ldots,n .$$

Therefore there are the slack variables $\Delta_k > 0 \quad (k=1,\ldots,n)$ that

$$\mu'^{1*} y_k^1 + \mu'^{c1*}\beta^* y_k^c + \mu'^{2*} y_k^2 + \mu'^{c2*}\left(1-\beta^*\right) y_k^c + \mu_1'^* + \mu_2'^* - \left(\tilde{v}^{1*} x_k^1 + \tilde{v}^{c1*}\alpha^* x_k^c + \tilde{v}^{2*} x_k^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_k^c\right) + \Delta_k 1_1 y_k^1 = 0,\ k=1,\ldots,n,$$

where 1_1 is a vector that all the components are one and number of its elements are as the number of components $y_k^1 \quad (k=1,\ldots,n)$. Taking $\Delta = min_{k=1,\ldots,n}\{\Delta_k\}$. So $\Delta > 0$ and $\Delta \le \Delta_k \quad (k=1,\ldots,n)$.

Therefore, the above equation will be as follows:

$$\mu'^{1*} y_k^1 + \mu'^{c1*}\beta^* y_k^c + \mu'^{2*} y_k^2 + \mu'^{c2*}\left(1-\beta^*\right) y_k^c + \mu_1'^* + \mu_2'^* - \left(\tilde{v}^{1*} x_k^1 + \tilde{v}^{c1*}\alpha^* x_k^c + \tilde{v}^{2*} x_k^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_k^c\right) + \Delta 1_1 y_k^1 \le 0,\ k=1,\ldots,n$$

$$\Rightarrow \left(\mu'^{1*} + \Delta 1_1\right) y_k^1 + \mu'^{c1*}\beta^* y_k^c + \mu'^{2*} y_k^2 + \mu'^{c2*}\left(1-\beta^*\right) y_k^c + \mu_1'^* + \mu_2'^* - \left(\tilde{v}^{1*} x_k^1 + \tilde{v}^{c1*}\alpha^* x_k^c + \tilde{v}^{2*} x_k^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_k^c\right) \le 0,\ k=1,\ldots,n.$$

On the other hand

$$\left(\mu'^{1*} + \Delta 1_1\right), \mu'^{c1*}, \mu'^{2*}, \mu'^{c2*}, \tilde{\mu}^{1*}, \tilde{\mu}^{c1*}, \tilde{\mu}^{2*}, \tilde{\mu}^{c2*}, \tilde{v}^{1*}, \tilde{v}^{c1*}, \tilde{v}^{2*}, \tilde{v}^{c2*}, \tilde{\mu}_1^*, \tilde{\mu}_2^*, \mu_1'^*, \mu_2'^*, \alpha^*, \beta^*$$

construct a feasible solution for model (7). The objective function value for this solution is larger than the optimal solution of problem (7), which is a contradiction. This contradiction is created since it was assumed that all the constraints (2-7) are strictly smaller than zero. Therefore contra-positive assumption is invalid and the proof holds true.

Theorem 5:
Model (7) calculates the relative performance of the two-components of DMUs.

Proof. If $\mu'^{i*}, \mu'^{ci*}, \tilde{v}^{i*}, \tilde{v}^{ci*}, \tilde{\mu}^{i*}, \tilde{\mu}^{ci*}, \tilde{\mu}_i^*, \mu_i'^*, \alpha^*, \beta^* \quad (i=1,2)$ are the optimal multipliers of model (7), according to the previous theorem

$$\max_{k=1,\ldots,n}\left\{\mu'^{1*} y_k^1 + \mu'^{c1*}\beta^* y_k^c + \mu'^{2*} y_k^2 + \mu'^{c2*}\left(1-\beta^*\right) y_k^c + \mu_1'^* + \mu_2'^* - \left(\tilde{v}^{1*} x_k^1 + \tilde{v}^{c1*}\alpha^* x_k^c + \tilde{v}^{2*} x_k^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_k^c\right)\right\} = 0$$

$$\Rightarrow \max_{k=1,\ldots,n}\left\{\frac{\mu'^{1*} y_k^1 + \mu'^{c1*}\beta^* y_k^c + \mu'^{2*} y_k^2 + \mu'^{c2*}\left(1-\beta^*\right) y_k^c + \mu_1'^* + \mu_2'^*}{\tilde{v}^{1*} x_k^1 + \tilde{v}^{c1*}\alpha^* x_k^c + \tilde{v}^{2*} x_k^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_k^c}\right\} = 1$$

So,

$$\mu'^{1*} y_o^1 + \mu'^{c1*}\beta^* y_o^c + \mu'^{2*} y_o^2 + \mu'^{c2*}\left(1-\beta^*\right) y_o^c + \mu_1'^* + \mu_2'^* = \frac{\mu'^{1*} y_o^1 + \mu'^{c1*}\beta^* y_o^c + \mu'^{2*} y_o^2 + \mu'^{c2*}\left(1-\beta^*\right) y_o^c + \mu_1'^* + \mu_2'^*}{1}$$

$$= \frac{\mu'^{*1} y_o^1 + \mu'^{*c1} \beta^* y_o^c + \mu'^{*2} y_o^2 + \mu'^{*c2}\left(1-\beta^*\right) y_o^c + \mu_1'^* + \mu_2'^*}{\tilde{v}^{1*} x_o^1 + \tilde{v}^{c1*} \alpha^* x_o^c + \tilde{v}^{2*} x_o^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_o^c} = \frac{\dfrac{\mu'^{*1} y_o^1 + \mu'^{*c1} \beta^* y_o^c + \mu'^{*2} y_o^2 + \mu'^{*c2}\left(1-\beta^*\right) y_o^c + \mu_1'^* + \mu_2'^*}{\tilde{v}^{1*} x_o^1 + \tilde{v}^{c1*} \alpha^* x_o^c + \tilde{v}^{2*} x_o^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_o^c}}{1}$$

$$= \frac{\dfrac{\mu'^{*1} y_o^1 + \mu'^{*c1} \beta^* y_o^c + \mu'^{*2} y_o^2 + \mu'^{*c2}\left(1-\beta^*\right) y_o^c + \mu_1'^* + \mu_2'^*}{\tilde{v}^{1*} x_o^1 + \tilde{v}^{c1*} \alpha^* x_o^c + \tilde{v}^{2*} x_o^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_o^c}}{\max\limits_{k=1,\ldots,n}\left\{\dfrac{\mu'^{1*} y_k^1 + \mu'^{c1*} \beta^* y_k^c + \mu'^{2*} y_k^2 + \mu'^{c2*}\left(1-\beta^*\right) y_k^c + \mu_1'^* + \mu_2'^*}{\tilde{v}^{1*} x_k^1 + \tilde{v}^{c1*} \alpha^* x_k^c + \tilde{v}^{2*} x_k^2 + \tilde{v}^{c2*}\left(1-\alpha^*\right) x_k^c}\right\}}$$

Therefore, the optimal value of the objective function of problem (7) is equal to the relative efficiency of DMU_o .

Noting that the model (7) is a nonlinear model and model (9) is a linear form which is equivalent to it. So, calculating the relative efficiency of two-component DMUs can be used in model (9).

Calculating the relative efficiency of components:

To obtain the efficiency of components under VRS assumption, at first, we solve model (9) and obtain the relative efficiency of each unit and then we add it as a constraint to problem and assume the priority components in terms of the decision maker (here, the first and second components, respectively), the values of the relative efficiency of the component is calculated. If θ_o^* is the optimal value of problem (9), the constraint $\mu^1 y_o^1 + \mu^{c1} y_o^c + \mu^2 y_o^2 + \mu^{c2} y_o^c + \mu_1 + \mu_2 = \theta_o^*$

is added to the problem. To obtain the relative efficiency of component one in the VRS assumption, a fractional programming model can be obtained which is as follows:

$$\max \frac{\dfrac{\tilde{\mu}^1 y_o^1 + \tilde{\mu}^{c1} y_o^c + \tilde{\mu}_1}{\tilde{v}^1 x_o^1 + \tilde{v}^{c1} x_o^c}}{\max\limits_{k=1,\ldots,n}\left\{\dfrac{\tilde{\mu}^1 y_k^1 + \tilde{\mu}^{c1} y_k^c + \tilde{\mu}_1}{\tilde{v}^1 x_k^1 + \tilde{v}^{c1} x_k^c}\right\}},$$

$$\begin{aligned}
s.t \quad & \mu'^1 y_o^1 + \mu'^{c1} y_o^c + \mu'^2 y_o^2 + \mu'^{c2} y_o^c + \mu_1' + \mu_2' = \theta_o^*, \\
& \tilde{v}^1 x_o^1 + \tilde{v}^{c1} x_o^c + \tilde{v}^2 x_o^2 + \tilde{v}^{c2} x_o^c = 1, \\
& \mu'^1 y_k^1 + \mu'^{c1} y_k^c + \mu'^2 y_k^2 + \mu'^{c2} y_k^c + \mu_1' + \mu_2' - (\tilde{v}^1 x_k^1 + \tilde{v}^{c1} x_k^c + \tilde{v}^2 x_k^2 + \tilde{v}^{c2} x_k^c) \le 0, \quad k = 1,\ldots,n, \\
& \tilde{\mu}^1 y_k^1 + \tilde{\mu}^{c1} y_k^c + \tilde{\mu}_1 - (\tilde{v}^1 x_k^1 + \tilde{v}^{c1} x_k^c) \le 0, \quad k = 1,\ldots,n, \\
& \tilde{\mu}^2 y_k^2 + \tilde{\mu}^{c2} y_k^c + \tilde{\mu}_2 - (\tilde{v}^2 x_k^2 + \tilde{v}^{c2} x_k^c) \le 0, \quad k = 1,\ldots,n, \\
& \mu'^1 \ge 0, \mu'^2 \ge 0, \mu'^{c1} \ge 0, \mu'^{c2} \ge 0,\ \tilde{\mu}^1 \ge 0, \tilde{\mu}^2 \ge 0, \tilde{\mu}^{c1} \ge 0, \tilde{\mu}^{c2} \ge 0, \tilde{v}^1 \ge 0, \tilde{v}^2 \ge 0, \tilde{v}^{c1} \ge 0, \tilde{v}^{c2} \ge 0, \\
& \tilde{\mu}_1, \tilde{\mu}_2, \mu_1', \mu_2' \ \ free,
\end{aligned} \tag{10}$$

The first constraint of the problem states that the relative efficiency DMU_o must be equal to the obtained amount by solving model (9). With the change of variables similar to the previous section, the linear equivalent model is generated as:

$$\begin{aligned}
max \quad & \tilde{\mu}^1 y_k^1 + \tilde{\mu}^{c1} y_k^c + \tilde{\mu}_1, \\
s.t \quad & \overline{\overline{v}}^1 x_o^1 + \overline{\overline{v}}^{c1} x_o^c = 1, \\
& \tilde{\mu}^1 y_o^1 + \tilde{\mu}^{c1} y_o^c + \tilde{\mu}_1 - \left(\overline{\overline{v}}^1 x_o^1 + \overline{\overline{v}}^{c1} x_o^c\right) \le 0,\ k = 1,\ldots,n, \\
& \mu'^1 y_o^1 + \mu'^{c1} y_o^c + \mu'^2 y_o^2 + \mu'^{c2} y_o^c + \mu_1' + \mu_2' = \theta_o^* h, \\
& \tilde{v}^1 x_o^1 + \tilde{v}^{c1} x_o^c + \tilde{v}^2 x_o^2 + \tilde{v}^{c2} x_o^c = h, \\
& \mu'^1 y_k^1 + \mu'^{c1} y_k^c + \mu'^2 y_k^2 + \mu'^{c2} y_k^c + \mu_1' + \mu_2' - (\tilde{v}^1 x_k^1 + \tilde{v}^{c1} x_k^c + \tilde{v}^2 x_k^2 + \tilde{v}^{c2} x_k^c) \le 0, \quad k = 1,\ldots,n, \\
& \tilde{\mu}^1 y_k^1 + \tilde{\mu}^{c1} y_k^c + \tilde{\mu}_1 - (\tilde{v}^1 x_k^1 + \tilde{v}^{c1} x_k^c) \le 0, \quad k = 1,\ldots,n,
\end{aligned}$$

$$\tilde{\mu}^2 y_k^2 + \tilde{\mu}^{c2} y_k^c + \tilde{\mu}_2 - (\tilde{v}^2 x_k^2 + \tilde{v}^{c2} x_k^c) \le 0, \qquad k = 1,\dots,n,$$
$$\mu'^1 \ge 0, \mu'^2 \ge 0, \mu'^{c1} \ge 0, \mu'^{c2} \ge 0, \tilde{\mu}^1 \ge 0, \tilde{\mu}^2 \ge 0, \tilde{\mu}^{c1} \ge 0, \tilde{\mu}^{c2} \ge 0, \tilde{v}^1 \ge 0, \tilde{v}^2 \ge 0, \tilde{v}^{c1} \ge 0, \tilde{v}^{c2} \ge 0,$$
$$\bar{\bar{v}}^1 \ge 0, \bar{\bar{v}}^{c1} \ge 0, h \ge 0, \tilde{\mu}_1, \tilde{\mu}_2, \mu'_1, \mu'_2 \ \ free \qquad (11)$$

The first component has a higher priority than the second component, so model (11) is solved and the optimal value is achieved. If φ_o^* is the above optimal value, then

$$\tilde{\mu}^1 y_o^1 + \tilde{\mu}^{c1} y_o^c + \tilde{\mu}_1 = \varphi_o^*,$$

as a constraint is added to the problem and the following model to obtain the relative efficiency of the second component, based on the overall efficiency and efficiency of the first component is presented as:

$$\max \ \tilde{\mu}^2 y_o^2 + \tilde{\mu}^{c2} y_o^c + \tilde{\mu}_2,$$
$$s.t \quad \bar{\bar{v}}^2 x_o^2 + \bar{\bar{v}}^{c2} x_o^c = 1,$$
$$\tilde{\mu}^2 y_o^2 + \tilde{\mu}^{c2} y_o^c + \tilde{\mu}_2 - \left(\bar{\bar{v}}^2 x_o^2 + \bar{\bar{v}}^{c2} x_o^c\right) \le 0, \ k = 1,\dots,n,$$
$$\tilde{\mu}^1 y_o^1 + \tilde{\mu}^{c1} y_o^c + \tilde{\mu}_1 = \varphi_o^*,$$
$$\bar{\bar{v}}^1 x_o^1 + \bar{\bar{v}}^{c1} x_o^c = t,$$
$$\tilde{\mu}^1 y_o^1 + \tilde{\mu}^{c1} y_o^c + \tilde{\mu}_1 - \left(\bar{\bar{v}}^1 x_o^1 + \bar{\bar{v}}^{c1} x_o^c\right) \le 0, \ k = 1,\dots,n,$$
$$\mu'^1 y_o^1 + \mu'^{c1} y_o^c + \mu'^2 y_o^2 + \mu'^{c2} y_o^c + \mu'_1 + \mu'_2 = \theta_o^* h,$$
$$\tilde{v}^1 x_o^1 + \tilde{v}^{c1} x_o^c + \tilde{v}^2 x_o^2 + \tilde{v}^{c2} x_o^c = h,$$
$$\mu'^1 y_k^1 + \mu'^{c1} y_k^c + \mu'^2 y_k^2 + \mu'^{c2} y_k^c + \mu'_1 + \mu'_2 - (\tilde{v}^1 x_k^1 + \tilde{v}^{c1} x_k^c + \tilde{v}^2 x_k^2 + \tilde{v}^{c2} x_k^c) \le 0, \quad k = 1,\dots,n,$$
$$\tilde{\mu}^1 y_k^1 + \tilde{\mu}^{c1} y_k^c + \tilde{\mu}_1 - (\tilde{v}^1 x_k^1 + \tilde{v}^{c1} x_k^c) \le 0, \quad k = 1,\dots,n,$$
$$\tilde{\mu}^2 y_k^2 + \tilde{\mu}^{c2} y_k^c + \tilde{\mu}_2 - (\tilde{v}^2 x_k^2 + \tilde{v}^{c2} x_k^c) \le 0, \qquad k = 1,\dots,n,$$
$$\mu'^1 \ge 0, \mu'^2 \ge 0, \mu'^{c1} \ge 0, \mu'^{c2} \ge 0, \tilde{\mu}^1 \ge 0, \tilde{\mu}^2 \ge 0, \tilde{\mu}^{c1} \ge 0, \tilde{\mu}^{c2} \ge 0, \tilde{v}^1 \ge 0, \tilde{v}^2 \ge 0, \tilde{v}^{c1} \ge 0, \tilde{v}^{c2} \ge 0,$$
$$\bar{\bar{v}}^1 \ge 0, \bar{\bar{v}}^2 \ge 0, \bar{\bar{v}}^{c1} \ge 0, \bar{\bar{v}}^{c2} \ge 0, h \ge 0, t \ge 0, \tilde{\mu}_1, \tilde{\mu}_2, \mu'_1, \mu'_2 \ \ free \qquad (12)$$

Numerical example:

In this section, an example including 10 units with a two-component structure is presented. Before this, we refer to example provided in Table 1. We saw that the score efficiency of units was less than unity based on the proposed method in the literature of two-component DEA. Now, we evaluate them by using the proposed method in this paper. As we see in the last column of Table 1, two DMUs are evaluated as efficient. To compare two units A and B, we have same common input for two units, and DMU A produces a better output related to DMU B in the first component and this matter is inverse in the second component. Therefore, it is reasonable to tell that two DMUs are not comparable. The result is evident based on the efficiency obtained by our proposed method, while score efficiency of units using the previous method concludes B has a better performance than A.

Here, we examine the suggested method for evaluating ten two–component units. Each of these units using 5 inputs, produce four outputs. The share of each component from the inputs of system is two inputs and one remaining input is shared between the two components. Finally, each of the two components, produce two independent outputs. The data of problem are summarized in the following Table (2).

Table 2: Data of 10 units with two-component structure

DMU	x_1^1	x_2^1	x_1^2	x_2^2	x^c	y_1^1	y_2^1	y_1^2	y_2^2
1	8098	3.42	2.51	109	58	7462	65.27	1491	381
2	8095	4.42	4.84	211	62	3122	60.91	666	256
3	15522	4.52	5.16	211	147	3109	61.24	555	260
4	15932	6.74	5.58	225	180	6318	97.52	61807	363
5	16555	5.25	6.25	242	154	4117	71.74	651	350
6	17441	9.53	6.09	44	238	4144	81.17	1821	85
7	14541	7.51	5.30	230	177	3114	62.20	715	240
8	19655	5.35	5.75	282	173	9639	98.63	925	189
9	18379	5.77	3.17	66	181	6031	92.02	1681	243
10	6931	3.36	2.66	214	172	6836	92.02	1437	410

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Table 3: The relative efficiency of two-component DMUs

Efficiency of component 2	Efficiency of component 1	Efficiency of two-component units	DMU
1.000000	1.000000	1.000000	1
0.9345339	0.9913284	0.9902783	2
0.497000	0.7423761	0.7413281	3
1.000000	1.000000	1.000000	4
0.4357522	0.6410886	0.6402160	5
1.000000	0.5384174	1.000000	6
0.4727990	0.5273496	0.5392857	7
0.4357637	1.000000	1.000000	8
1.000000	0.8592700	1.000000	9
1.000000	1.000000	1.000000	10

Based on models (9), (11) and (12) respectively, relative efficiency of 10 units and the relative efficiency of the first and second components are achieved and that their information is saved in Table 3. As it can be seen, units 1, 4, 6, 8, 9 and 10 are relatively efficient. Among the relative efficient units, the relative efficiency of the first and second components of units 1, 4 and 10 is equal to number one. In other words, units 1, 4 and 10 are efficient in terms of the perspective of each of the components and also in terms of the perspective of the overall performance. Although units 6 and 9 have the relative performance equal to one, but they are not relatively efficient based on the first component. Also, unit 8 is efficient in the relative performance and in the first component, while it is not efficient in terms of the second component. Other units, i.e., units 2, 3, 5 and 7 are inefficient in all and the first and second components.

Conclusion:

In this paper, a method is proposed in order to provide a solution for the relative efficiency of two-component DMUs under VRS. Validity of the method was demonstrated by the theorems. In this model, at first, the relative performance was obtained and then added as a constraint to the problem and in the following parts based on the relative performance of each unit, component with the highest of priority were evaluated.

In this way, the relative efficiency of the components based on the relative efficiency and the priorities of decision makers was given. This method has the advantage that in those the poor fields (components) can be identified and the ability of corrective action for more desirable performance of DMU is possible. Every single estimating return to scale of components of every unit and the relationships among them with returns to scale of two-component units can be performed based on the proposed model. The authors suggest the introduction of the oriented models corresponding the proposed multiple models in this paper for further studies in order to provide the benchmark units for the inefficient units.

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