Advances in Environmental Biology, 8(11) Special 2014, Pages: 597-605

AENSI Journals

Advances in Environmental Biology

ISSN-1995-0756 EISSN-1998-1066

Journal home page: http://www.aensiweb.com/aeb.html

Some properties of harmonic graphs

[1]**Ahmad Salehi Zarrin Ghabaei and** [2]**Shahroud Azami**

[1]*Parand Branch, Islamic Azad University, Tehran, Iran*
[2]*Parand Branch, Islamic Azad University, Tehran, Iran*

ARTICLE INFO

Article history:
Received 25 April 2014
Received in revised form
8 May 2014
Accepted 20 May 2014
Available online 16 August 2014

Key words:
Graph, Eigenvalue, Eigenvector, Adjacency matrix, Planar graph.
2010 MSC: 15A18, 05C50, 05C10.

ABSTRACT

Let G be a graph with n vertices and m edges and where is the degree of the vertex .A graph G is called -harmonic whenever d(G) is one of the eigenvectors of G. Eigenvalues of the adjacent matrix of a graph is important subject in graph theory and its applications. On the other hand, some operators on graphs have fundamental roles graph theory. In this paper, we consider harmonic graphs and study some graph operators such as graph products, graph sums, graph complement, linear harmonic graphs on its. In final, we study the planar harmonic graph in partial case.

To Cite This Article: Ahmad Salehi Zarrin Ghabaei and Shahroud Azami, Some properties of harmonic graphs. ***Adv. Environ. Biol., 8(11),*** *597-605, 2014*

INTRODUCTION

Suppose that $G = (V(G), E(G))$ is a simple graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. Let $d(v_i)$ be the degree (number of first neighbors) of the vertex $v_i, i = 1, 2, ..., n$. Vertex of degree k is called a k-vertex. Vertex of degree zero is called an isolated vertex. Vertex of degree one is called a pendant. Let $d(G) = (d(v_1), d(v_2), ..., d(v_n))^T$. The number of k-vertex denoted by n_k and we have

$$\sum_{k\geq 0} n_k = n, \tag{1}$$

$$\sum_{k\geq 0} kn_k = 2m. \tag{2}$$

Definition 1.1. The adjacency matrix $A(G) = [a_{ij}]$ is the $n \times n$ matrix for which $a_{ij} = 1$ if $v_i v_j \in E(G)$ and $a_{ij} = 0$ otherwise. Eigenvalues and eigenvalue vectors of adjacency matrix A(G) is called eigenvalues and eigenvalue vectors of graph G.

Definition 1.2. A graph G is said to be harmonic if there exists a constant λ, such that

$$\lambda d(v_i) = \sum_{v_i v_j \in E(G)} d(v_j), \qquad i = 1, 2, ..., n, \tag{3}$$

in other words

$$A(G)d(G) = \lambda d(G), \tag{4}$$

therefore, graph G is harmonic if and only if d(G) is one of its eigenvectors. These graphs are called λ-harmonic.

Example 1.3. A λ-regular graph is a λ-harmonic graph.

Remark 1.4. Equation 3 implies that λ is a rational number and from Equation 4 and that graph eigenvalues are not proper fractions, it follows that λ must be an integer.

Corresponding Author: Shahroud Azami, Parand Branch, Islamic Azad University, Tehran, Iran.
E-mail: shahrood78@aut.ac.ir

Advances in Environmental Biology, 8(11) Special 2014, Pages: 597-605

Summing up the Equation 3 from i = 1 to i = n, observing that each summed $d(v_j)$ is counted $d(v_j)$ times, hence

$$\sum_{v\in V(G)} d(v)(d(v)-\lambda)=0, \tag{5}$$

i.e.,

$$\sum_{k\geq 0} k(k-\lambda)n_k=0, \tag{6}$$

which are necessary, but not sufficient.

Lemma 1.5. [2]
a) Let H be a graph obtained from G by adding to it an arbitrary number of isolated vertices. Then H is harmonic if and only if G is harmonic.
b) Any graph without isolated vertices is λ-harmonic if and only if all its components are λ-harmonic.
c) Let G be a connected λ-harmonic graph with the greatest eigenvalue λ of multiplicity one. If $m>0$ then $\lambda\geq 1$ and equality occurs if and only if $G=K_2$.

According to the above lemma it is enough to study the harmonics of connected non-regular graphs. For any positive integer λ there is a unique connected λ-harmonic that is a tree denoted by T_λ in the following manner ([3]). T_λ has $\lambda^3-\lambda^2+\lambda+1$ vertices, of which one vertex is a $(\lambda^2-\lambda+1)$-vertex, $\lambda^2-\lambda+1$ vertices are λ-vertices and $(\lambda-1)(\lambda^2-\lambda+1)$ vertices are pendant.

In this paper we would like to answer to the followings questions.
1) Is the complement of a harmonic graph a harmonic graph?
2) Is the product of harmonic graphs a harmonic graph?
3) Are harmonic graphs planar?

1.1 Graphs products:

We now introduce four fundamental graph products: the Cartesian product, the direct product, the strong product, and the lexicographic product. In each case, the product of graphs G and H is another graph whose vertex set is the Cartesian product $V(G)\times V(H)$. However, each product has different rules for adjacencies.

The Cartesian product of G and H is a graph, denoted by $G\square H$, whose vertex set is $V(G)\times V(H)$. Two vertices (g,h) and (g',h') are adjacent precisely if $g=g'$ and $hh'\in E(H)$, or $gg'\in E(G)$ and $h=h'$. Thus,

$$V(G\square H)=\{(g,h)\mid g\in V(G)\quad and\quad h\in V(H)\},$$
$$E(G\square H)=\{(g,h)(g',h')\mid g=g',\ hh'\in E(H),\ \ or\ \ gg'\in E(G),\ h=h'\}$$

The graphs G and H are called factors of $G\square H$.

The direct product of G and H is the graph, denoted by $G\times H$, whose vertex set is $V(G)\times V(H)$, and for which vertices (g,h) and (g',h') are adjacent if $gg'\in E(G)$ and $hh'\in E(H)$. Thus,

$$V(G\times H)=\{(g,h)\mid g\in V(G)\quad and\quad h\in V(H)\},$$
$$E(G\times H)=\{(g,h)(g',h')\mid gg'\in E(G)\quad and\quad hh'\in E(H)\}.$$

Other names for the direct product that have appeared in the literature are tensor product, Kronecker product, cardinal product, relational product, cross product, conjunction, weak direct product, Cartesian product, product, or categorical product.

The strong product of G and H is the graph denoted by $G\boxtimes H$, and denoted by

$$V(G\boxtimes H)=\{(g,h)\mid g\in V(G)\,and\ h\in V(H)\},$$
$$E(G\boxtimes H)=E(G\square H)\bigcup E(G\times H).$$

Occasionally one also encounters the names strong direct product or symmetric composition for the strong product. Note that $G\square H$ and $G\times H$ are subgraphs of $G\boxtimes H$. For clarity, the edges of the subgraph $G\square H$ are drawn in bold. Finally, the lexicographic product of graphs G and H is the graph $G\circ H$ with

$$V(G\circ H)=\{(g,h)\mid g\in V(G),\ h\in V(H)\},$$
$$E(G\circ H)=\{(g,h)(g',h')\mid gg'\in E(G),\ \text{or}\ \ g=g'\ \text{and}\ \ hh'\in E(H)\}.$$

Advances in Environmental Biology, 8(11) Special 2014, Pages: 597-605

Graph products were studied in [1].

2 Some operators on harmonic graphs:
2.1 Complement of a harmonic graph:

Definition 2.1. Let $G=(V(G),E(G))$ be a graph with vertex set $\{v_1,v_2,...,v_n\}$. $G^c=(V(G^c),E(G^c))$ called the complement graph of G, in which $V(G^c)=V(G)$ and $(v_i,v_j)\in E(G^c)$ if and only if $v_iv_j\notin E(G)$.

In this section we investigate that under what circumstances the complement of a harmonic graph can be a harmonic graph. Let G be a λ-harmonic graph, if G^c is a μ-harmonic graph, then for all $v\in V(G)$ we have

$$\mu d_{G^c}(v)=\sum_{uv\in E(G^c)} d_{G^c}(u)=\sum_{uv\notin E(G)}(n-1-d_{G^c}(u))=\sum_{uv\notin E(G)}(n-1)-\sum_{uv\notin E(G)} d_G(u), \tag{7}$$

hence

$$\mu(n-1-d_G(v))=(n-1)(n-1-d_G(v))-\sum_{uv\notin E(G)} d_G(u), \tag{8}$$

on the other hand

$$2m=\sum_{u\in V(G)} d_G(u)=\sum_{uv\in E(G)} d_G(u)+\sum_{uv\notin E(G)} d_G(u)+d_G(v)=(\lambda+1)d_G(v)+\sum_{uv\notin E(G)} d_G(u). \tag{9}$$

Therefore

$$\sum_{uv\notin E(G)} d_G(u)=2m-(\lambda+1)d_G(v), \tag{10}$$

we can rewrite Equation 8 as follow:

$$(\mu-n+1)(n-1-d_G(v))=-2m+(\lambda+1)d_G(v). \tag{11}$$

If G is a non regular λ-harmonic graph, then, it has at least two vertices w_0 and w_1 of distinct degrees. Hence , using 11 for these two vertices, implies that

$$\begin{cases}(\mu-n+1)(n-1-d_G(w_0))=-2m+(\lambda+1)d_G(w_0),\\ (\mu-n+1)(n-1-d_G(w_1))=-2m+(\lambda+1)d_G(w_1).\end{cases} \tag{12}$$

These two equations produce the following:

$$(\mu-n+1)(d_G(w_1)-d_G(w_0))=(\lambda+1)(d_G(w_0)-d_G(w_1)), \tag{13}$$

so

$$(\mathrm{d_G(w_1)}-\mathrm{d_G(w_0)})(\mu-\mathrm{n}+\lambda+2)=0,$$

thus $\mu-n+\lambda+2=0$ and

$$\mu=n-\lambda-2. \tag{14}$$

Replacing 14 in 11, results in

$$2m=(\lambda+1)(n-1), \tag{15}$$

therefore n or λ must be odd. According to the above description, we have

Lemma 2.2. Let $G=(V(G),E(G))$ be a λ-harmonic graph with $V(G)|=n$ and $|E(G)|=m$ and there is a constant number μ so that 19 is satisfied for every $v\in V(G)$. Then G^c is a μ-harmonic graph.

Lemma 2.3. Let $G=(V(G),E(G))$ be a non-regular λ-harmonic graph with $V(G)|=n$ and $|E(G)|=m$ and its complement G^c be a μ-harmonic graph. Then we have

1) $(\mu-n+1)(n-1-d_G(v))=-2m+(\lambda+1)d_G(v)$ for all $v\in V(G)$,
2) $(n-1-d_G(v))\,|\,(\lambda+1)d(v)-2m$ for all $v\in V(G)$,
3) $\mu=n-\lambda-2$,
4) $2m=(\lambda+1)(n-1)$,
5) $(n-1)(\lambda-1)\geq\lambda^2-2\lambda+2$.

Proof. Statements 1 to 4 can be obtained directly applying the same argument as above description, but a proof the final statement, in [2] show that $m-n+1\geq \lambda^2-2\lambda+2$.

Lemma 2.4. The complement of every connected regular λ-harmonic graph is a connected regular $(n-1-\lambda)$- harmonic graph.

Corollary 2.5. The complement of a tree T_λ is not harmonic graph for each $\lambda\neq 1$.

Proof. In trees we have $m=n-1$ so using Lemma 2.3 implies that $2m=(\lambda+1)(n-1)=(\lambda+1)m$ hence $\lambda=1$. This is a contradiction.

Corollary 2.6. Let $G=(V(G),E(G))$ be a connected non-regular λ-harmonic graph with $|V(G)|=n$ and $|E(G)|=m$ and n and λ are even. Then the complement of G is not a harmonic graph.

Proof. According to 15, n and λ are odd, that is not the case.

Corollary 2.7. Let $G=(V(G),E(G))$ be a connected non-regular λ-harmonic graph with $|V(G)|=n$ and $|E(G)|=m$ and there are arithmetic numbers r and s such that $m=m+s$ and $2(s+r)$ is not divisible by $(n-1)$, then G^c is not a harmonic graph.

Proof. If G^c be a harmonic graph then Lemma 2.3 results in that $2m=(\lambda+1)(n-1)$, then $2(m+s)=(\lambda+1)(n-1)$ therefore $2(m+s)$ is divisible by $(n-1)$, this is a contradiction.

Corollary 2.8. Let G and G^c be harmonic graphs. Then 2c is divisible by $(n-1)$ where $c=m-n+1$.

Proof. Using Lemma 2.3 we have $2m=(\lambda+1)(n-1)$ and $2(\mathrm{c}+(\mathrm{n}-1))=(\lambda+1)(\mathrm{n}-1)$ and this implies that $2c=(\lambda-1)(n-1)$ and so 2c is divisible by $(n-1)$.

Remark 2.9. In [2] λ-harmonic graphs with $c=1,2,3,4$ are characterized where complement of their non-regular graphs are not harmonic graphs.

2.2 Linear graph of a harmonic graph:

Definition 2.10. Let $G=(V(G),E(G))$ be a λ- harmonic graph with $V(G)=\{v_1,v_2,...,v_n\}$ and $E(G)=\{e_1,e_2,...,e_n\}$. The linear graph of G denote by L_G where its vertices set is edges set of the graph G and in L_G two vertices e_i,e_j are adjacent if e_i,e_j have a common vertex. Let G be a connected regular λ-harmonic then its linear graph is a $\mu=2\lambda-2$ harmonic graph.

Theorem 2.11. Let $G\neq T_2$ be a connected nonregular λ-harmonic such that it has a vertex of degree one, then its linear graph is not a harmonic graph.

Proof. Let L_G be a λ-harmonic graph. Suppose that $e_i=u_iv_i$ is a arbitrary vertex of L_G.

$$\begin{aligned}\mu d(e_i) &= \sum_{e_j \text{ is associated with } e_i \text{ in a vertex}} d(e_j) \qquad (16)\\ &= \sum_{e_j \text{ is associated with } e_i \text{ in vertex } u_i} d(e_j) + \sum_{e_j \text{ is associated with } e_i \text{ in vertex } v_i} d(e_j)\\ &= \sum_{uu_i\in E(G),u_i\neq v_i} (d_G(u_i)+d_G(u)-2) + \sum_{vv_i\in E(G),u_i\neq v_i} (d_G(v_i)+d_G(v)-2),\end{aligned}$$

hence

$$\begin{aligned}\mu(d_G(u_i)+d_G(v_i)-2) &= (-2+d_G(u_i))(d_G(u_i)-1)+\lambda d_G(u_i)-d_G(v_i)+(-2+d_G(v_i))(d_G(v_i)-1)+\lambda d_G(v_i)-d_G(u_i) \qquad (17)\\ &= (-2+d_G(u_i))(d_G(u_i)-1)+(-2+d_G(v_i))(d_G(v_i)-1)+(\lambda-1)(d_G(u_i)+d_G(v_i)).\end{aligned}$$

Since G has a vertex u of degree one, therefore this vertex has an adjoint vertex v of degree λ, so, from 17, we have $\mu=2\lambda-1$. On the other hand, $d(v)=\lambda$, hence v is adjacent to a vertex a other than vertex u say w. Considering the edge vw and 17 implies:

$$\mu(\lambda+d_G(w)-2)=(-2+d_G(w))(d_G(w)-1)+(\lambda-2)(\lambda-1)+(\lambda-1)(d_G(w)+\lambda),$$

since $\mu=2\lambda-1$, so $(d_G(w))^2-(\lambda+3)d_G(w)+(\lambda+2)=0$ therefore $d_G(w)=1$ or $d_G(w)=\lambda+2$.

So all adjacent vertices of v are of degree 1 or $\lambda+2$. Suppose the number of vertices of degree λ adjacent to v is k. If $k=0$, then

$$\lambda^2=\lambda d_G(v)=\sum_{uv\in E(G)} d_G(u)=d_G(v)=\lambda,$$

hence $\lambda=0$ or $\lambda=1$, which is a contradiction, so $k\neq 0$. On the other hand we have

$$\lambda^2=\lambda d(v)=\sum_{(u,v)\in E(G)} d(u)=k(\lambda+2)+\lambda-k,$$

so $k=\dfrac{\lambda(\lambda-1)}{\lambda+1}$. k is a natural number, therefore $\lambda(\lambda-1)$ is divisible by $(\lambda+1)$ but $gcd(\lambda,\lambda-1)=1$, hence $\lambda-1$ is divisible by $(\lambda+1)$ which is a contradiction. Thus the theorem is established.

Corollary 2.12. The linear graph T_λ is not a harmonic graph.

2.3 The sum of harmonic graphs:

The sum of G and H is a graph, denoted by G + H, whose vertex set is $V(G)\cup V(H)$ and two vertices u and v are adjacent if $u\in V(G)$ and $v\in V(H)$ or $uv\in E(H)$, or $uv\in E(G)$. Thus

$$V(G+H)=V(G)\cup V(H),$$
$$E(G\cup H)=\{uv \mid u\in V(G) \text{ and } v\in V(H) \text{ or } uv\in E(H), \text{ or } uv\in E(G)\}.$$

Let $G=(V(G),E(G))$ and $G'=(V(G'),E(G'))$ with $|V(G)|=n,|V(G')|=n',|E(G)|=m,|E(G')|=m'$ and G,G' are λ and λ' connected harmonic graphs, respectively. Let $H=G+G'$. To check whether a graph H is harmonic or not, suppose that $x\in V(G)$, we have

$$\sum_{ux\in E(H)} d_H(u)=\sum_{u\in V(G),ux\in E(G)} d_H(u)+\sum_{u\in V(G)} d_H(u)$$
$$=\sum_{ux\in E(G)}(d_G(u)+n')+\sum_{u\in V(G)}(d_{G'}(u)+n)$$
$$=\lambda d_G(x)+n'd_G(x)+2m'+nn'.$$

Similarly, if $x\in V(G')$ then

$$\sum_{ux\in E(H)} d_H(u)=\lambda' d_{G'}(x)+nd_{G'}(x)+2m+nn'.$$

Corollary 2.13. Let $G=(V(G),E(G))$ and $G'=(V(G'),E(G'))$ with $|V(G)|=n,|V(G')|=n',|E(G)|=m,|E(G')|=m'$ and G,G' are λ and λ' connected regular graphs, respectively. Let $H=G+G'$. If $\lambda+n'=\lambda'+n$ then H is a $(\lambda+n')$-harmonic graph.

Corollary 2.14. Let $G=(V(G),E(G))$ and $G'=(V(G'),E(G'))$ with $|V(G)|=n,|V(G')|=n',|E(G)|=m,|E(G')|=m'$ and G,G' are λ and λ' connected harmonic graphs, respectively. Let $H=G+G'$. If $\lambda+n'=\lambda'+n, 2m=n\lambda, 2m'=n\lambda'$ then H is a $(\lambda+n')$-harmonic graph.

3 Harmonic graph products:

3.1 The Cartesian product of harmonic graph:

Lemma 3.1. The Cartesian product of the non-regular harmonic graphs is not a harmonic graph.

Proof. Let $G=(V(G),E(G))$ and $G'=(V(G'),E(G'))$ be λ and λ' harmonic graphs respectively. Suppose that $H=G\square G'$ is a μ- harmonic graph. For all $x\in V(G)$ and $y'\in V(G')$ we have

$$\mu d_H(x,y')=\sum_{(u,v)(x,y')\in E(H)} d_H(u,v)=\sum_{ux\in E} d_H(u,y')+\sum_{y'v\in E'} d_H(x,v),$$

therefore

$$\mu(d_G(x)+d_{G'}(y')) = \sum_{ux\in E(G)} (d_G(u)+d_{G'}(y')) + \sum_{y'v\in E(G')} (d_G(x)+d_{G'}(v)),$$

hence,

$$\mu = \frac{\lambda d_G(x)+\lambda' d_{G'}(y')+2d_G(x)d_{G'}(y')}{d_G(x)+d_{G'}(y')} \qquad \forall x\in V(G),\ \forall y'\in V(G'). \tag{18}$$

Since G and G' are connected non-regular harmonic, so, each has at least two vertices $x,y\in V(G)$ and $x',y'\in V(G')$ of distinct degrees. Using 18 for vertex (x,x') and (x,y'), we have

$$\frac{\lambda d_G(x)+\lambda' d_{G'}(y')+2d_G(x)d_{G'}(y')}{d_G(x)+d_{G'}(y')} = \frac{\lambda d_G(x)+\lambda' d_{G'}(x')+2d_G(x)d_{G'}(x')}{d_G(x)+d_{G'}(x')},$$

it results in

$$\lambda = \lambda' + 2d_G(x). \tag{19}$$

Similarly, for vertices (x,y') and (y,y') we have

$$\lambda' = \lambda + 2d_{G'}(y') \tag{20}$$

19 and 20 implies $d_G(x)+d_{G'}(y')=0$ so $d_G(x)=0$ and $d_{G'}(y')=0$. This is a contradiction. Hence the Cartesian product of non-regular harmonic graphs is not a harmonic graph.

Lemma 3.2. The Cartesian product of regular harmonic graphs is a harmonic graph.

Proof. Let $G=(V(G),E(G))$ and $G'=(V(G'),E(G'))$$ be λ and λ' connected regular graphs respectively. then $H=G\square G'$ is a $\lambda+\lambda'$ regular graph and therefore it is a $\lambda+\lambda'$ harmonic graph.

Corollary 3.3. 1) Let G be a harmonic graph. Then using 18, $G\square G$ is a harmonic graph if and only if G is a regular graph.

2) The Cartesian product of a connected regular harmonic graph and a connected non regular harmonic graph is not a harmonic graph.

3.2 The lexicographic product of harmonic graph:

Lemma 3.4. The lexicographic product of non-regular harmonic graphs is not a harmonic graph.

Proof. Let $G=(V(G),E(G))$ and $G'=(V(G'),E(G'))$ be λ and λ' harmonic graph respectively. Suppose that $H=G\circ G'$ is a μ-harmonic graph. Let $|V(G)|=n, |V(G')|=n'$. For all $x\in V(G)$ and $y'\in V(G')$ we have $d_H(x,y')=n'd_G(x)+d_{G'}(x')$ and

$$\begin{aligned}\mu d_H(x,y') &= \sum_{(u,v)(x,y')\in E(H)} d_H(u,v)\\ &= \sum_{ux\in E(G)} (n'd_G(u)+d_{G'}(v)) + \sum_{y'v\in E(G')} (n'd_G(x)+d_{G'}(v))\\ &= n'\lambda d_G(x)+d_G(x)d_{G'}(v)+n'd_G(x)d_{G'}(y')+\lambda' d_{G'}(y'),\end{aligned}$$

The above relation satisfies for all $v\in V(G')$, and it is impossible. Hence the lexicographic product of non-regular harmonic graphs is not a harmonic graph.

Lemma 3.5. The lexicographic product of the regular harmonic graphs is a harmonic graph.

Corollary 3.6. 1) Let G be a harmonic graph. Then $G\circ G$ is a harmonic graph if and only if G is a regular graph.

2) The lexicographic product of a connected regular harmonic graph and a connected non regular harmonic graph is not a harmonic graph.

3.3 The strongly product of harmonic graph:

Lemma 3.7. The strongly product of connected non-regular harmonic graphs is not a harmonic graph.

Proof. Let $G=(V(G),E(G))$ and $G'=(V(G'),E(G'))$ be λ and λ' harmonic graph respectively. Suppose that $H=G\boxtimes G'$ is a μ-harmonic graph. For all $x\in V(G)$ and $y'\in V(G')$ we have $d_H(x,y')=d_G(x)d_{G'}(y')+d_G(x)+d_{G'}(y')$ and

Advances in Environmental Biology, 8(11) Special 2014, Pages: 597-605

$$\mu d_H(x,y') = \sum_{(u,v)(x,y')\in E(H)} d_H(u,v) = \sum_{ux\in E(G)} d_H(u,y') + \sum_{y'v\in E(G')} d_H(x,v) + \sum_{ux\in E(G),y'v\in E(G')} d_H(u,v)$$
$$= (\lambda\lambda' + 2\lambda + 2\lambda' + 2)d_G(x)d_{G'}(y') + \lambda d_G(x) + \lambda' d_{G'}(y'),$$

Since G and G' are connected non-regular harmonic, so, each has at least two vertices $x, y \in V(G)$ and $x', y' \in V(G')$ of distinct degrees. Using 44 for vertices (x, x') and (x, y'), we have

$$\begin{cases} \mu(d_G(x)d_{G'}(x') + d_G(x) + d_{G'}(x')) = (\lambda\lambda' + 2\lambda + 2\lambda' + 2)d_G(x)d_{G'}(x') + \lambda d_G(x) + \lambda' d_{G'}(x') \\ \mu(d_G(x)d_{G'}(y') + d_G(x) + d_{G'}(y')) = (\lambda\lambda' + 2\lambda + 2\lambda' + 2)d_G(x)d_{G'}(y') + \lambda d_G(x) + \lambda' d_{G'}(y'). \end{cases}$$

Subtracting these two equations yields

$$\mu d_G(x) + 1 = (\lambda\lambda' + 2\lambda + 2\lambda' + 2)d_G(x) + \lambda', \tag{21}$$

Similarly, for vertex (y, x') and (y, y') we have

$$\mu d_G(y) + 1 = (\lambda\lambda' + 2\lambda + 2\lambda' + 2)d_G(y) + \lambda', \tag{22}$$

21 and 22 implies $\mu = \lambda\lambda' + 2\lambda + 2\lambda' + 2$ and this is a contradiction. Hence the strongly product of non-regular harmonic graphs is not a harmonic graph.

Lemma 3.8. The strongly product of regular harmonic graphs is a harmonic graph.

3.4 The direct product of harmonic graphs:

Let $G = (V(G), E(G))$ and $G' = (V(G'), E(G'))$ be λ and λ' connected harmonic graphs, respectively. Let $H = G \times G'$. We have $N_H(x,y) = \{(u,v) \mid u \in N_G(x), v \in N_{G'}(y)\}$, hence $d_H(x,y) = d_G(x)d_{G'}(y)$ and

$$\sum_{(u,v)\in N_H(x,y)} d_H(u,v) = \sum_{u\in N_G(x), v\in N_{G'}(y)} d_G(u)d_{G'}(v) = \sum_{v\in N_{G'}(y)} \Big(d_{G'}(v) \sum_{u\in N_G(x)} d_G(u)\Big)$$
$$= \sum_{v\in N_{G'}(y)} \lambda d_{G'}(v)d_G(x) = \lambda\lambda' d_{G'}(y)d_G(x) = \lambda\lambda' d_H(x,y),$$

therefore H is a $\lambda\lambda'$ harmonic graph.

4 Planar harmonic graphs:

Theorem 4.1. There is no planar connected regular harmonic graph with maximum degree $\Delta = n-1$.

Proof. Assume that the statement is not established. Let G be a connected planar non-regular λ-harmonic graph with n vertices, m edges and $\Delta = n-1$. Suppose that u is a vertex of G that $d(u) = \Delta = n-1$. Hence

$$\lambda d(u) = \sum_{(u,v)\in E(G)} d_G(v) = 2m - d_G(u),$$

so $2m = (\lambda+1)(n-1)$. Since $\Delta = n-1$, so in λ-harmonic graph G there isn't a vertex of degree one, because in λ-harmonic graphs any vertex of degree one is adjacent to a vertex of degree λ, also, each vertex of degree greater than one is adjacent to a vertex with a maximum degree . Unless, $\lambda = n-1$, that in this case, the graph is regular and it contradicts the assumption.

For all vertices $v \neq u$, we have

$$\lambda d_G(v) = n - 1 + \sum_{wv\in E(G), w\neq u} d_G(w) \geq n - 1 + 2(d_G(v) - 1),$$

therefore

$$(\lambda - 2)d_G(v) \geq n - 3. \tag{23}$$

Note that if $\lambda = 2$ then T_2 is the only connected non-regular 2-harmonic graph that n = 7 and $\Delta = 3 \neq 6$, so $\lambda = 2$ is not satisfied in lemma's circumstances. Thus, without the loss of generality, we may assume that $\lambda \geq 3$. Using 23, we have

$$d_G(v) \geq \frac{n-3}{\lambda - 2}, \qquad \forall v \in V(G), v \neq u,$$

then $\lambda d_G(u) = \sum_{v \in V(G)-\{u\}} d_G(v)$, implies that

$$\lambda^2 - 2\lambda + 2 \geq n-1,$$

on the other hand in [2] it is shown that $2m - 2n + 2 \geq \lambda^2 - 2\lambda + 2$. Since graph G is planar therefore $3n - 6 \geq m$, then

$$4n - 10 \geq \lambda^2 - 2\lambda + 2 \geq n - 1, \tag{24}$$

also $(\lambda+1)(n-1) = 2m \leq 6n - 12$ implies that $\lambda \leq 5 - \frac{6}{n-1}$. Hence $3 \leq \lambda \leq 5 - \frac{6}{n-1}$. According to 24, for n, λ we have the following eight cases;

Case 1) n = 4, λ = 3.

In this case we have

$$3(3) = \lambda d_G(u) = d_G(v_1) + d_G(v_2) + d_G(v_3).$$

So at least one of the vertices v_1, v_2 and v_3 has degree greater than 3, which is impossible.

Case 2) n = 5, λ = 3.

Since graph G is non-regular, so G has vertex of degree 2 or 3.

a) If $v \in V(G)$, $d_G(v) = 2$, then we assume that the adjacent vertices of v are u and w such that $d_G(u) = \Delta = 4$. So $3d_G(v) = 4 + d_G(w)$ and this implies that $d_G(w) = 2$. Suppose x and y are two other vertices, then

$$3(4) = \lambda d_G(u) = d_G(v) + d_G(w) + d_G(x) + d_G(y) = 4 + d_G(x) + d_G(y),$$

it implies that $d_G(x) + d_G(y) = 8$, this is a contradiction.

b) If $v \in V(G)$, $d_G(v) = 3$, then we assume that the adjacent vertices of v are u, x and y such that $d(u) = \Delta = 4$. So $3d(v) = 4 + d(x) + d(y)$ implies that $d_G(x) + d_G(y) = 5$. Hence $(d_G(x), d_G(y)) = (1,4)$ or (4, 1) or (2, 3) or (3,2). This is a contradiction (graph G can not have vortices of degree 1, 2).

Case 3) n = 6, λ = 3.

Since graph G is non-regular, so G has vertex of degree 2 or 3 or 4.

a) If $v \in V(G)$, $d_G(v) = 2$, then we assume that the adjacent vertices of v are u and x such that $d(u) = \Delta = 5$. So $3d_G(v) = 5 + d_G(x)$ and this implies that $d_G(x) = 1$. This is a contradiction and graph doesn't have any vertex of degree 2.

b) If $v \in V(G)$, $d_G(v) = 3$, then we assume that the adjacent vertices of v are u, x and y such that $d_G(u) = \Delta = 5$. So $3d_G(v) = 5 + d_G(x) + d_G(y)$ and this implies that $d_G(x) + d_G(y) = 4$. This is a contradiction and graph doesn't have any vertex of degree 3.

c) If $v \in V(G)$, $d_G(v) = 4$, then we assume that the adjacent vertices of v are u, x, y and z such that $d_G(u) = \Delta = 5$. So $3d_G(v) = 5 + d_G(x) + d_G(y) + d_G(z)$ and this implies that $d_G(x) + d_G(y) + d_G(z) = 7$. This is a contradiction (graph doesn't have any vertex of degree $1,2,3$).

Case 4) n = 7, λ = 4.

Since graph G is non-regular, then G has vertex of degree 2 or 3 or 4 or 5.

a) If $v \in V(G)$, $d_G(v) = 2$, then we assume that the adjacent vertices of v are u and x such that $d_G(u) = \Delta = 6$. So $4d_G(v) = 6 + d_G(x)$ and this implies that $d_G(x) = 2$. Hence $4d_G(u) = 4 + d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4)$ results in $d_G(v_1) + d_G(v_2) + d_G(v_3) + d_G(v_4) = 20$. This is a contradiction and graph doesn't have any vertex of degree 2.

Advances in Environmental Biology, 8(11) Special 2014, Pages: 597-605

b) If $v \in V(G)$, $d_G(v) = 3$, then we assume that the adjacent vertices of v are u, x and y such that $d_G(u) = \Delta = 6$. So $4d_G(v) = 6 + d_G(x) + d_G(y)$ and this implies that $d_G(x) + d_G(y) = 6$ From a) we have $d_G(x) = d_G(y) = 3$ and $4d_G(u) = 9 + d_G(v_1) + d_G(v_2) + d_G(v_3)$ results in $d_G(v_1) + d_G(v_2) + d_G(v_3) = 15$, this is a contradiction and graph doesn't have any vertex of degree 3.

c) If $v \in V(G)$, $d_G(v) = 4$ or5, then similar to cases a), b) this is a contradiction.

Case 5) n = 8, λ = 4.

$2m = (\lambda + 1)(n - 1) = 35$ this is a contradiction.

Case **6)** n = 9, λ = 4.

Similar to case 4 this case is impossible.

Case 7) n = 10, λ = 4.

$2m = (\lambda + 1)(n - 1) = 45$ this is a contradiction.

Case 8) n = 11, λ = 4.

Similar to case 4 this case impossible.

Remark 4.2. 1) The graph T_λ is planar.

2) The 1-harmonic and 2-harmonic graphs are planar.

Lemma 4.3. There is exactly four regular maximal planar λ-harmonics graph.

Proof. In a maximal planar graph $G = (V(G), E(G))$ with $|V(G)| = n$ and $|E(G)| = m$, we have $m = 3n - 6$. Now, If G is a regular λ-harmonics graph, then $n\lambda = 2m = 6n - 12$ and this implies that $12 = n(6 - \lambda)$, hence

$$(6 - \lambda, n) \in \{(1,12), (2,6), (3,4), (4,3), (6,2), (12,1)\}$$

cases $6 - \lambda = 6, 12$ are impossible, and in other cases graph is planar and harmonic.

REFERENCES

[1] Borovicanin, B., S. Grunewald, I. Gutman, M. Petrovic, 2003. Harmonic graphs with small number of cycles. Discrete Mathematics, 265: 31-44.

[2] Grunewald, S., 2002. Harmonic tree, Appl. Math. Lett. 15.

[3] Hammack, R., W. Imrich, S. Klavzar, 2011. Handbook of Product Graphs, Second Edition (Discrete Mathematics and Its Applications), United States of America on acid-free paper.