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New Approximations of Fuzzy Numbers and Their Applications

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ABSTRACT

Background: The point and interval operators of the fuzzy numbers can be used to transform the fuzzy comparison matrices to the point and interval comparison matrices in the fuzzy AHP, and the fuzzy data envelopment analysis (FDEA). The percentiles, skewness, kurtosis and the correlation of fuzzy numbers can play an important role for the better recognition of the patterns in the fuzzy models. Some of these operators can be used for ranking fuzzy numbers, to estimate of the parameters of statistical models in a fuzzyenvironment. **Objective:** To find the new approximations of fuzzy numbers and their applications. **Results:** Compared the new point and interval approximations of fuzzy numbers with the previously approximations of the fuzzy numbers. **Conclusion:** In this paper, we have introduced the weighted interval approximation of a fuzzy number, which is an extension of the works of Grzegorzewski. The new fuzzy interval-valued means and the interval-valued modes of a fuzzy number are proposed. Furthermore, based on the percentiles of fuzzy numbers the new point and interval approximations of a fuzzy number are presented. Also a type of the correlation coefficient is mentioned for comparing interval approximations. These crisp approximately operators of fuzzy numbers are simple, suitable and play an important role in the fuzzy problems

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INTRODUCTION

Fuzzy set theory proposed by Zadeh (1965) allows us to process and transform imprecise information effectively and flexibly. Fuzzy numbers play a significant role among all fuzzy sets since the predominate representation of information is numeric. Representing fuzzy numbers by suitable intervals is an interesting and important problem. An interval approximation of a fuzzy number may have many useful applications. By using such a representation, it is possible to apply in fuzzy number approaches some results derived in the field of interval number analysis. For example, it may be applied to a comparison of fuzzy numbers by using the order relations defined on the set of interval numbers. Also, the problem of interval and fuzzy number comparison is of perennial interest because of its direct relevance in modeling and optimization of real world processes.

Approximation of fuzzy quantities by scalar or crisp intervals has been studied by many researchers, Dubois and Prade (1978,1987,2000), Ma et al. (2000), Carlsson and Fulle'r (2001), Grzegorzewski (2002), Bodjanova (2005,2006) . Approximations of fuzzy numbers by scalars derived from the descriptive statistics are used in statistical analysis of crisp numerical data. Bodjanova (2006) introduced the percentile characterizations of alpha-cut of a fuzzy number, and obtained some interval approximations by special alpha-cuts. In the nature of fuzzy samples, many fundamental statistics, such as mean, median, and mode and etc., are useful measurements in illustrating some characteristics for the population distribution. These statistical parameters can be quickly computed from a set of data and their basic information has been widely employed in applications. Each statistics has its special application. For example, when we want to investigate people's opinions or consensus on a certain public issue, the use of mode or median will be more proper than that of mean. However, traditional statistics reflect the results from a two valued logic opinion. To investigate the population, people's opinions or the complexity of a subjective event more accurately, it is suggested that we should use fuzzy logic. Especially, when we want to know the public opinion on the environmental pollution, fuzzy statistic provides a powerful research tool. However, we want to present some of the new definitions and techniques on the fuzzy numbers, so that can be applied for statistical inferences on fuzzy data. These approximations are interesting when comparing fuzzy numbers, and can be applied for solving many problems in fuzzy systems. For example, systems of linear and nonlinear equations with fuzzy parameters are relevant to many practical problems arising in structure mechanics, electrical engineering, finance, economics and physics; solving such equations depends

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on interval parameters. Therefore, from the practical point of view, these approximations of fuzzy numbers can help to researchers for better recognition of patterns in the fuzzy models.

In this paper, we introduce a weighted interval approximation of an arbitrary fuzzy number such that in applications, the weight function can be chosen according to the actual situation by decision makers. In subsection 3.2 we present fuzzy intervals that are obtained from fuzzy sample data based on expert opinion. In section 4, the median value and interval-valued median of fuzzy numbers are recalled. In subsection 4.1, we introduce the percentiles and percentile intervals of a fuzzy number that are new and different from the percentiles definition of fuzzy numbers by Bodjanova (2006). In section 5, we present a new mode value of a trapezoidal fuzzy number and it is extended to a family of fuzzy numbers. In section 6, some the approximations of a fuzzy number are compared through three theorems. In section 7, the correlation coefficient between two fuzzy intervals is introduced, and several applications are mentioned for further research in future. Conclusion is given in section 8.

2. Preliminarily:

Let *R* be the set of all real numbers. We assume that for all $x \in R$, the fuzzy number *A* can be expressed in the following form:

$$A(x) = \begin{cases} A_L(x) & x\epsilon[a,b], \\ 1 & x\epsilon[a,b], \\ A_R(x) & x\epsilon[a,b], \\ 0 & therwise. \end{cases} \tag{1}$$

Where *a*, *b*, *c*, *d* are real numbers such that $a < b \leq c < d, A_L$ is a real valued increasing and right continuous function and *A* is a real valued decreasing and left continuous function. Notice that (1) is in fact an L-R fuzzy number with strictly monotone shape function, as proposed by Dubois and Prade in 1981 and also described by Dubois and Prade (1978). Each fuzzy number *A* described by (1) has the following $\alpha -$ level sets $(\alpha - cuts)[A]_\alpha = [A_L^{-1}(\alpha), A_R^{-1}(\alpha)] = [a_1(\alpha), a_2(\alpha)]$, for all $\alpha \in (0,1], [A]_0 = supp\ (A)$, the support of $A, supp(A) = \{x \in R | A(x) > 0\}$ is bounded. For a fuzzy number A, its $\alpha - cut$ set, $[A]_\alpha = [a_1(\alpha), a_2(\alpha)]$ is a closed interval, where $a_1(\alpha) = in\ f\{x | A(x) \geq \alpha\}$. If the left and right sides of fuzzy number *A* are strictly monotone, obviously, $a_1(\alpha), a_2(\alpha)$ are inverse functions of *AL* and *AR*, respectively. The space of all fuzzy numbers will be denoted by *F (R)* and *P(R)* denotes a family of all closed intervals on the real line that, the members of *P(R)* are subset of *R*. Let *A* be a fuzzy number with the following shape functions

$$A_L(x) = (\frac{x-a}{b-a})^n, A_R(x) = (\frac{d-x}{d-c})^n$$

Where u>0 , we denote this subfamily of fuzzy numbers by $F_n^T(R) \subset F(R)$ or $A = (a,b,c,d)_n$, (see Fig.1), If n=1, then *A=(a, b, c, d)* is called a trapezoidal fuzzy number. If $n \neq 1$, the fuzzy number a concentration of *A*. If 0<n<1, then *A** is a dilation of *A*.

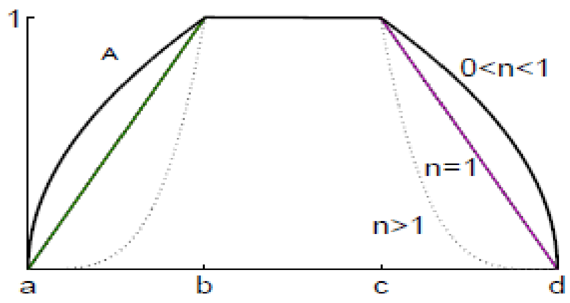

Fig. 1: Fuzzy number A = (a, b, c, d)$_n$.

Note that the fuzzy number *A* with the membership function *A(x)* (which has an unbounded support) is defined as:

$$A(x) = \begin{cases} A_L(x) & x \leq b, \\ 1 & x \epsilon [b,c], \\ A_R(x) & x \geq c. \end{cases} \quad (2)$$

2.1. Mean value and interval-valued mean:

In this subsection, we first recall several intervals of a fuzzy number *A*, and then the new approximations of *A* introduced in two subsections.

Definition 1:

(Carlson and Fuller 2001) The possibilistic mean value and the interval valued possibilistic mean of the fuzzy number *A* are defined as

$$M(A) = \int_0^1 a\big(a_1(a) + a_2(a)\big)da, \quad (3)$$

$$IM(A) = [M_1(A), M_2(A)] = [2\int_0^1 a a_1(a)da, 2\int_0^1 a a_2(a)da]\,. \quad (4)$$

In [13] showed that (3) and (4) are the nearest point and interval of the fuzzy number *A*, respectively. Dubois and Prade (1987) have introduced the expected interval of fuzzy number *A* as

$$EI(A) = [E_1(A), E_2(A)] = \left[\int_0^1 a_1(a)da, \int_0^1 a_2(a)da,\right]$$

The middle point of EI (A) is $E(A) = \int_0^1 \frac{a_1(a)+a_2(a)}{2} d\alpha$, that is called the mean value of fuzzy number A.

2.2. f- weighted mean and interval-valued mean:

In this subsection, we use the weighted distance between two fuzzy numbers to investigate the point and interval approximations of arbitrary fuzzy numbers. Thus, we have to use an operator $C: F(R) \to P(R)$which transforms a fuzzy number into a family of the closed intervals on the real line. Given the fuzzy number A and its $\gamma - cut$ set $A_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$, our aim is to find an interval of fuzzy number A as $c_d^f(A) = [C_L, C_U]$, where C_L is its lower bound and C_U upper bound. $c_d^f(A)$ is called weighted interval approximation of the fuzzy number *A* with respect to the weighed distance d, that defined by

$$d\left(A, C_d^f(A)\right) = \sqrt{\int_0^1 \frac{1}{2}[f(\gamma)\left(\underline{a}(\gamma) - C_L\right)^2 d\gamma + \int_0^1 f(\gamma)(\overline{a}(\gamma) - C_U)^2 d\gamma\Big]}\,. \quad (5)$$

Where $c_d^f(A) = [C_L, C_U]$ is an interval of the support function, i.e. $[c_d^f(A)]_\gamma = [C_L, C_U]$. And the function $f\,;[0,1] \to R$ is nonnegative, monotonic increasing, and $\int_0^1 f(\gamma) = 1$. In fact, the relationship (5) is a type of the weighted distance between the endpoints of its level sets and two points of the support function fuzzy number *A*. The close interval $c_d^f(A) = [C_L, C_U]$ is the nearest weighted interval to A based on (5) (more see [13]), where

$$C_d^f(A) = [\int_0^1 f(\gamma)\underline{a}(\gamma)d\gamma,\ \int_0^1 f(\gamma)\overline{a}(\gamma)d\gamma]. \quad (6)$$

Note that different methods to find interval approximations of a fuzzy set are introduced. The easiest way is to substitute a fuzzy number either by its support C_0= *supp (A)* or by its core *C=core (A)*, and generally, $C_\gamma = \{x\epsilon R | A(x) \geq \gamma\}$ for any $\gamma \in [0,1]$ is an interval approximation of fuzzy sets, namely, C_0 is probably the best known and it is the most popular in practice operator. However, all this operators have quite a

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natural interpretation and sometimes a very unpleasant drawback-the lack of continuity. Here the nearest weighted interval approximation of fuzzy number is obtained so that it has good summarized information of a fuzzy number; specially, when weighting function f is usable and changeable in the fuzzy AHP problems.

Remark 1:

We also can de ne the nearest weighted interval approximation of a fuzzy number in the family of all left-sided fuzzy numbers (FLS(R)) and the family of all right-sided fuzzy numbers (FRS(R)), such that the nearest weighted interval approximation of the left-sided fuzzy number A is:

$$C_d^f(A) = \left[\int_0^1 f(\gamma)\underline{a}(\gamma)d\gamma, +\infty\right); \quad A\epsilon F_{LS}(R), \tag{7}$$

and the nearest weighted interval approximation of the right-sided fuzzy number A is:

$$C_d^f(A) = \left(-\infty, \int_0^1 f(\gamma)\underline{a}(\gamma)d\gamma\right]; \quad A\epsilon F_{RS}(R). \tag{8}$$

In applications, the weight function $f(\gamma)$ can be chosen according to the actual situation; hence, decision-maker's can select different functions to approximate a fuzzy number by the weighted intervals, more see Saeidifar (2009, 2011). In fact, our weighted interval approximation is more flexibility than the interval approximation introduce by Grzegorzewski (2002), because we can change the weighting function in the different applications.

Remark 2:

If f = 1 then the operator $C_d^f(A): F(R) \to R(R)$ is the operator $C_d(A)$, that defined by Grzegorzewski (2002).

Definition 2:

Let A be a fuzzy number with $[A]_\gamma = [\underline{a}(\gamma), \overline{a}(\gamma)]$ and $f(\gamma)$ is a weight function. The middle point of the interval $C_d^f(A) = [C_L, C_U]$ is defined as a weighted mean value of the fuzzy number *A*,

$$\overline{M}_f(A) = \frac{C_L + C_U}{2} = \int_0^1 f(\gamma)\,\frac{\underline{a}(\gamma)+\overline{a}(\gamma)}{2}d\gamma. \tag{9}$$

Example 1:

Let A = (a, b, c, d)n be a fuzzy number, Then

$$EI(A) = [E_1(A), E_2(A)] = \left[a(b-a)\frac{n}{n+1}, d-(d-c)\frac{n}{n+1}\right],$$

$$IM(A) = [M_1(A), M_2(A)] = \left[a(b-a)\frac{2n}{2n+1}, d-(d-c)\frac{2n}{2n+1}\right],$$

$$E(A) = \frac{a+d}{2} + \frac{n}{2(n+1)}(b+c-a-d), M(A) = \frac{a+d}{2} + \frac{n}{2n+1}(b+c-a-d).$$

Example 2:

Let A be a fuzzy number with the following membership function and the weight functions $f_1(\gamma) = 2\gamma, f_2(\gamma) = 3\gamma^2$

$$A(x) = \exp\left[-\frac{(x-\theta)^2}{\sigma^2}\right], \quad x \in R \tag{10}$$

Where θ is the position of the peak relative to the universe, and $\sigma > 0$ standard deviation.

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We get $\gamma -$ cut set $[A]_\gamma = \left[\theta - \sigma\sqrt{-Ln(\gamma)}, \theta + \sigma\sqrt{-Ln(\gamma)}\right], for\ all\ \gamma\epsilon[0,1]$, and hence

$$C_L = \int_0^1 f_1(\gamma)\underline{a}(\gamma)d\gamma = 2\int_0^1 \gamma\left(\theta - \sigma\sqrt{-Ln(\gamma)}\right)d\gamma = \theta - \sqrt{\frac{\pi}{8}}\sigma,$$

$$C_U = \int_0^1 f_1(\gamma)\overline{a}(\gamma)d\gamma = 2\int_0^1 \gamma\left(\theta + \sigma\sqrt{-Ln(\gamma)}\right)d\gamma = \theta + \sqrt{\frac{\pi}{8}}\sigma,$$

I.e.

$$C_d^{f1}(A) = [\theta - \sqrt{\frac{\pi}{8}}\sigma, \theta + \sqrt{\frac{\pi}{8}}\sigma].$$

The operator $C_d^{f1}(A)$ is the nearest weighted interval approximation of the fuzzy number *A* with respect to the weighed metric d that given by (5). Also,

$$C_d^{f2}(A) = [\theta - \sqrt{\frac{\pi}{12}}\sigma, \theta + \sqrt{\frac{\pi}{12}}\sigma].$$

Example 3:

Consider the fuzzy number C with $\gamma -$cut set

$$[\underline{c}(\gamma), \overline{c}(\gamma)] = [-\sqrt{1-\gamma}, \sqrt{1-\gamma}],$$

and the weight function $f(\gamma) = 2\gamma$. Then our the weighted interval approximation is [-0.533, 0.533] and the interval approximation of Grzegorzewski (2002) is [-0.667, 0.667] (see Fig.2). Also the weighted point approximation is $\overline{M}_f(c) = 0$.

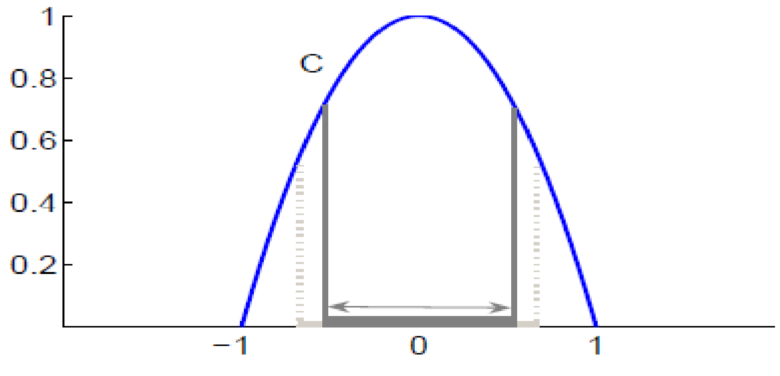

Fig. 2: Fuzzy number C and the two interval approximations.

Fig.2 shows that with change the weight function f the interval approximation is changed.

3. Fuzzy intervals based on expert opinion:

In this section we will to estimate the value for a certain parameter θ. First assume that we have only expert. Let θ_1 be the smallest possible value for θ, let θ_3 be the largest possible value for, and let θ_2 be the most likely value. We can ask the expert to give values for $\theta_1, \theta_2, \theta_3$ and we construct the triangular fuzzy estimator $\tilde{\theta}_1 = (\theta_1, \theta_2, \theta_3)$ for θ. Now suppose we have N expert. We still want to construct a triangular fuzzy estimate for $\tilde{\theta}_1 = (\theta_1, \theta_2, \theta_3)$. The easiest way to do this is to ask the experts for their $\theta_{1i}, \theta_{2i}, \theta_{3i}$, for all$1 \leq i \leq N$, and then take average of each component. For find the confidence interval we use from the intersection of confidence intervals for all fuzzy numbers $\tilde{\theta}_i$. Let $\theta_{1i} = \theta_1$ and $\theta_{3i} = \theta_3$ for any $1 \leq i \leq N$, this means that for all $\tilde{\theta}_i$ support functions are equals. Hence, the following interval and point estimators are introduced for the fuzzy parameter $\tilde{\tilde{\theta}}$.

$$EI\left(\tilde{\tilde{\theta}}\right) = \left[E_1\left(\tilde{\tilde{\theta}}\right), E_2\left(\tilde{\tilde{\theta}}\right)\right], \tag{11}$$

Where

$$E_1\left(\tilde{\tilde{\theta}}\right) = \max\{E_1\left(\tilde{\theta}\right), 1 \leq i \leq N\}, \tag{12}$$

$$E_2\left(\tilde{\tilde{\theta}}\right) = \min\{E_2(\tilde{\theta}_i), 1 \leq i \leq N[E_1(\tilde{\theta}_i), 1 \leq i \leq N\}\}, \tag{13}$$

$$E\left(\tilde{\tilde{\theta}}\right) = \frac{E_1\left(\tilde{\tilde{\theta}}\right)+E_2\left(\tilde{\tilde{\theta}}\right)}{2}. \tag{14}$$

Example 4:

Let $\tilde{\theta}_1 = (0,1,4), \tilde{\theta}_2 = (0,2,4), \tilde{\theta}_3 = (0,3,4)$ be three triangular fuzzy numbers (see Fig.3). We get the following table:

	EI	IM	E	M
$\tilde{\theta}_1$	[0.5,2.5]	[0.667,2.000]	1.5	1.333
$\tilde{\theta}_2$	[1.0,3.0]	[1.333.2.667]	2	2
$\tilde{\theta}_3$	[1.5,3.5]	[2.000,3.333]	2.5	2.667
$\tilde{\tilde{\theta}}$	[1.5,2.5]	[2.000.2.000]	2	2

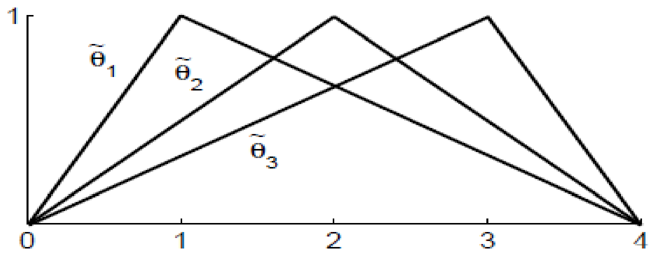

Fig. 3: Fuzzy numbers $\tilde{\theta}_1$, $\tilde{\theta}_2$,$\tilde{\theta}_3$.

When one of the methods to find the interval estimators is not proper, we apply another method, e.g. $IM\left(\tilde{\tilde{\theta}}\right) = [2,2]$, that cannot be as a suitable interval-valued mean. If we apply simple method of take the average of each component, and then we have $\tilde{\tilde{\theta}} = (0,2,4)$.

We can also apply other methods to find the interval estimators from the fuzzy sample data as follows:

$$EI^*\left(\tilde{\tilde{\theta}}\right) = \left[\frac{\sum_{i=1}^{N} E_1(\tilde{\theta}_i)A_{\tilde{\theta}_i}(E_1(\tilde{\theta}_i))}{\sum_{i=1}^{N} A_{\tilde{\theta}_i}(E_2(\tilde{\theta}_i))}, \frac{\sum_{i=1}^{N} E_2(\tilde{\theta}_i)A_{\tilde{\theta}_i}(E_2(\tilde{\theta}_i))}{\sum_{i=1}^{N} A_{\tilde{\theta}_i}(E_2(\tilde{\theta}_i))}\right]. \tag{15}$$

$$IM^*\left(\tilde{\tilde{\theta}}\right) = \left[\frac{\sum_{i=1}^{N} M_1(\tilde{\theta}_i)A_{\tilde{\theta}_i}\left(M_1(\tilde{\theta}_i)\right)}{\sum_{i=1}^{N} A_{\tilde{\theta}_i}\left(M_1(\tilde{\theta}_i)\right)}, \frac{\sum_{i=1}^{N} M_2(\tilde{\theta}_i)A_{\tilde{\theta}_i}\left(M_2(\tilde{\theta}_i)\right)}{\sum_{i=1}^{N} A_{\tilde{\theta}_i}\left(M_2(\tilde{\theta}_i)\right)}\right]. \tag{16}$$

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The above intervals are interval-valued estimators from the approximation intervals of the fuzzy sample data. For Example 3, we obtain

$$EI^*\left(\tilde{\tilde{\theta}}\right) = [1,3], \qquad IM^*\left(\tilde{\tilde{\theta}}\right) = [1.333,2.667].$$

Definition 3:

(Nguyen and Wu 2006) Let X be the universe set, and $\{F_{xi} = [a_i, b_i]; a_i, b_i \epsilon R, i = 1,2,\dots,n\}$ be a sequence of random fuzzy sample on X. Then the fuzzy sample mean value is defined as follows:

$$F\bar{x} = [\frac{1}{n}\sum_{i=1}^{n} a_i, \frac{1}{n}\sum_{i=1}^{n} b_i]. \tag{17}$$

Example 5:

Consider the fuzzy intervals $EI_1 = [0.5,2.5], EI_2 = [1,3], \ EI_3 = [1.5,3.5]$ from Example 4. Then the fuzzy sample mean value is $\overline{\mathrm{EI}} = [1,3]$.

Example 6:

Let the intervals $x_1 = [2,3], x_2 = [3,4], x_3 = [4,6], x_4 = [5,8], x_5 = [3,7], x_6 = [2,5]$ be the beginning salary for 6 new master graduated students, Then fuzzy sample mean for the beginning salary of the graduated students will be

$$F\bar{x} = [\frac{2+3+4+5+3+2}{6}, \frac{3+4+6+8+7+5}{6}] = [3.17,5.5]$$

Example 7:

Let the fuzzy time series$\{\tilde{x}_t\} = \{\tilde{x}_{t1}, \tilde{x}_{t2}, \dots, \tilde{x}_{tn}\}$, be the variation of a stock's value with n days (see Fig.4). We can compute the interval-valued means of the stock's value as

$$\{[x1_t, x2_t]\} = \{[x1_{t1}, x2_{t2}], [x1_{t2}, x2_{t2}], \dots, [x1_{tn}, x2_{tn}]\},$$

So

$$F\bar{x}_t = [\frac{1}{n}\sum_{i=1}^{n} x1_{ti}, \frac{1}{n}\sum_{i=1}^{n} x2_{tn}].$$

Note that $[x1_{ti}, x2_{ti}]$ are interval approximations from the fuzzy numbers $\tilde{\mathrm{x}}_t, i = 1,2,\dots,n.$

Now we shall de ne the fuzzy correlation between $\tilde{\mathrm{x}}_t, \tilde{\mathrm{x}}_{t+k}$ that is called the *k*th order autocorrelation of $\{\tilde{\mathrm{x}}_t\}$The sample estimate of this fuzzy autocorrelation, is calculated using the formula:

$$Fr_k = [\underline{r}_k, \bar{r}_k] = [\min\{r1_k, r2_k\}, \max\{r1_k, r2_k\}] \tag{18}$$

Where

$$r1_k = \frac{\sum_{i=1}^{i=n-k}(x1_{ti} - \overline{x1}_t)(x1_{ti} - \overline{x1}_t)}{\sum_{i=1}^{i=n-k}(x1_{ti} - \overline{x1}_t)^2},$$

$$r2_k = \frac{\sum_{i=1}^{i=n-k}(x2_{ti} - \overline{x2}_t)(x2_{ti} - \overline{x2}_t)}{\sum_{i=1}^{i=n-k}(x2_{ti} - \overline{x2}_t)^2},$$

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$$\overline{x1}_t = \frac{1}{n}\sum_{i=1}^{n} x1_{ti}, \overline{x2}_t = \frac{1}{n}\sum_{i=1}^{n} x2_{ti}.$$

Autocorrelations are used extensively in time series analysis and it is a correlation coefficient. The above autocorrelation function can be used for the two purposes, to detect non-randomness in fuzzy data and to identify an appropriate time series model if the fuzzy data are not random. When the autocorrelation is used to detect non-randomness, it is usually only the first (k=1) autocorrelation that is of interest. When the autocorrelation is used to identify an appropriate time series model, the autocorrelation are usually plotted for many lags (k=1, 2, ... , n-1). However, the Fr_k plays an important role in the estimation of fuzzy time's series models.

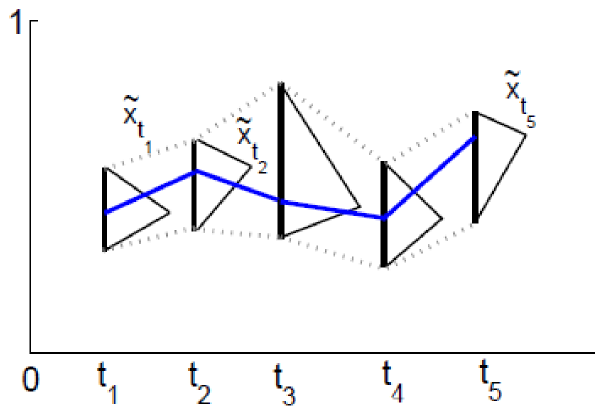

Fig. 4: Fuzzy time series.

In the Table 1 the interval-valued means of fuzzy numbers are summarized.

Table 1: The interval-valued means of fuzzy number *A*.

Previously	New
$C_d^1, EI(A), IM(A), F\overline{x}$	$C_d^f, EI^*(A), IM^*(A), Fr_k$

4. New interval approximations of fuzzy numbers:

In this section, first some the definitions and notations are recalled. Then we introduce a new and natural definition of the percentiles of fuzzy numbers. The median value of a fuzzy number was defined by Dubois and Prade (1987) as the numerical value from the support function, it that equally, divides the area under the membership function.

Definition 4:

The median value of a fuzzy number A is the numerical value *Me (A)* of the support function so that

$$\int_a^{Me(A)} A(x)dx = \frac{1}{2}\int_a^d A(x)dx. \tag{19}$$

Definition 5:

Let A be a fuzzy umber with the membership function *A(x)* so that $x_0 = \min\{x|A(x) = 1\}$ and $y_0 = \min\{x|A(x) = 1\}$. Then the interval $IMe = [Me_1(A), Me_2(A)]$, from the support function is defined as the interval-valued median of the fuzzy number A, where $Me_1(A)$ and $Me_2(A)$ obtain by solving

$$\int_a^{Me_1(A)} A(x)dx = \frac{1}{2}\int_a^{x_0} A(x)dx, \int_{yo}^{Me_2(A)} A(x)dx = \frac{1}{2}\int_{yo}^{d} A(x)dx. \tag{20}$$

For example, let A = (a, b, c, d)$_n$. Then

$$IMe = [Me_1(A), Me_2(A)] = \left[a + (b-a)2^{-1/(n+1)}, d - (d-c)2^{-1/(n+1)}\right].$$

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One can see that for the fuzzy number $A(a,b,c,d)_n, IMe = [A]_\alpha$, so that

$\alpha = 2^{-1/(n+1)} 2^{-1/(n+1)}$

This means that at least membership grade for the interval-valued median of the fuzzy number *A* is $2^{-1/(n+1)}$.

Fuzzy set *A* defined on the universal set X can be characterized by several numerical properties: The height, the cardinality and the width of *A*.

1. *The height:*

$$\mathrm{hgtA} = \sup_{x \epsilon X} A(x).$$

2. *The cardinality:*

When $supp\ A = \{x \epsilon X | A(x) > 0\}$ is a finite set, then the cardinality of A is

$$|A| = \sum_{x \epsilon suppA} A(x).$$

When $supp\ A$ is infinite, then $|A|$ is

$$|A| = \int_X A(x) dP(x), \tag{21}$$

Where X is a measurable set and P is a measure on X such that

$$\int_X \mathrm{dP(x)} = 1.$$

In this case, for $\alpha \epsilon (0,1]$ the cardinality of

$\alpha - cut[A]_\alpha = \{x \epsilon X \big| A(x) \geq \alpha\}\ is\ |A_\alpha| = \int_{A_\alpha} dP(x)$ and $|A_0| = \int_{supp\ A} dP(x)$.

Then the width of A is defined by

$$wdtA = \sup_{\alpha \epsilon [0,1]} \{|A_\alpha|\} = |A_0|.$$

If A is a fuzzy number then $hgtA = 1, wdtA = a_2(0) - a_1(0)$ and

$$|A| = \int_R A(x) dx = \int_0^1 (a_2(\alpha) - a_1(\alpha) d\alpha = S(A).$$

We now will introduce another interval for fuzzy numbers. For any $\gamma \in [0,1]$ and the function $\alpha(\gamma) \in (0, hgtA]$, we define

$$S_{\alpha(\gamma)}(A) = \int_0^{\alpha(\gamma)} [a_2(\alpha) - a_1(\alpha)] d\alpha, \tag{22}$$

Where $\alpha(\gamma)$ is a non-decreasing function in term $\gamma \in (0,1)$, and an alpha-level set of the fuzzy number *A*. Note that $S_{\alpha(\gamma)}(A)$ is the area under membership function and below the $\alpha(\gamma)$-level set of the fuzzy number *A* (see Fig.5). Therefore

$S_{\alpha(\gamma)}(A) = 0; if\ \alpha(\gamma) = 0, \quad S_{\alpha(\gamma)}(A) = S(A); if\ \alpha(\gamma) = 1.$

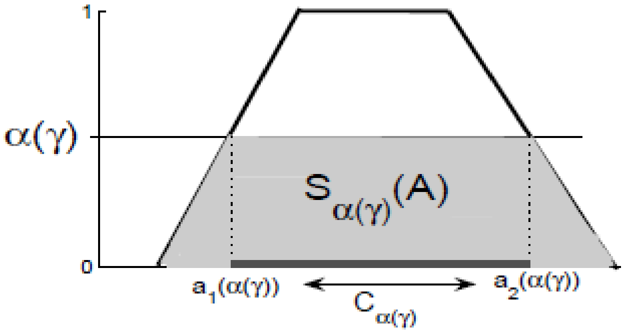

Fig. 5: Fuzzy number A and its interval approximation.

Definition 6:

Let A be a fuzzy set on the universal set X. The value $\alpha(\gamma) \in (0, hgtA]$ is called the $100\gamma\ th$ percentile alpha-level of *A* with respect to the cardinality of *A* if

$$\frac{S_{\alpha(\gamma)}(A)}{S(A)} = \gamma. \tag{23}$$

For given value $\gamma \in (0,1)$, we obtain the value $\alpha(\gamma)$. Then a percentile interval approximation of A is

$$C_{\alpha(\gamma)}(A) = [a_1(\alpha(\gamma)), a_2(\alpha(\gamma))]. \tag{24}$$

Notice: if $\gamma = 1$ then $\alpha(\gamma) = 1$, and so $C_{\alpha(\gamma)}(A) = C_1(A) = [b, c]$.

Theorem 1:

Let A be a fuzzy number with $[A]_\alpha = [a_1(\alpha), a_2(\alpha)]$, and α is a non-decreasing function and $0 \leq \gamma 1 \leq \gamma 2 \leq 1$.Then

$(i) S_{\alpha(\gamma 1)}(A) \leq S_{\alpha(\gamma 2)}(A).$

$(ii) C_{\alpha(\gamma 2)}(A) = [a_1(\alpha(\gamma 2)), a_2(\alpha(\gamma 2))] \subseteq [a_1(\alpha(\gamma 1)), a_2(\alpha(\gamma 1))] = C_{\alpha(\gamma 1)}(A).$

Proof. The proof is obvious (see Fig.5 and Fig.6).

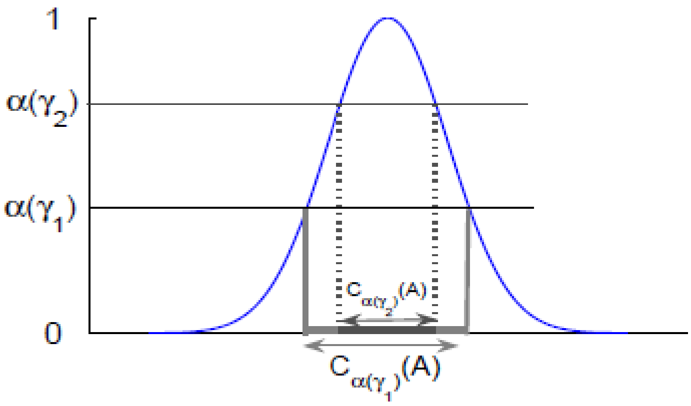

Fig. 6: Fuzzy number *A*, and the two interval approximations.

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Theorem 2:

Let $A \in F(R), \alpha(\gamma)$ be a uniform sequential from γ , and $\{0 < \gamma_m < 1, m = 1,2,\dots\}$ be an increasing sequential of real numbers such that

$$\lim_{m \to \infty} \gamma_m = 1.$$

Then

$$\lim_{m \to \infty} C_{\alpha(\gamma_m)} = \lim_{m \to \infty}[a_1(\alpha(\gamma_m)), a_2(\alpha(\gamma_m))] = [a_1(1), a_2(1)] = [b, c].$$

Proof:

Since $\gamma_m \to 1$ then $\alpha(\gamma_m) \to 1$. Therefore, by using (23) and (24) the theorem proves. This theorem shows that with increasing the value γ, the fuzzy interval tend to a crisp interval.

Example 8:

Let $A = (0, 1, 3, 4)$be a trapezoidal fuzzy number with $[A]_\alpha = [\alpha, 4 - \alpha]$ and $\gamma = 0.5$. Then we obtain

$$S_{\alpha(0.5)} = \int_0^\alpha (4 - \alpha - \alpha)d\alpha = 0.5 \times S(A) = 0.5 \int_0^1 (4 - \alpha - \alpha)d\alpha, \text{so}$$

$\alpha(0.5) = 0.4189$ and $C_{\alpha(0.5)} = [0.4189, 3.5811]$.

If $\gamma = 0.7$ then we get $\alpha(0.7) = 0.6216$ and $C_{\alpha(0.7)} = [0.6216, 3.3784]$.

Observe that the value alpha-level=0.4189 of A is correspond to 50th percentile with respect to the cardinality of A and $S_{\alpha(0.5)}$.

And clearly, $[0.6216, 3.3784] = C_{\alpha(0.7)} \subset C_{\alpha(0.5)} = [0.0.4189, 3.3.5811]$ (see Fig.7).

The above interval approximations of a fuzzy number are new and interesting such that can be applied for comparing of fuzzy number in different problems.

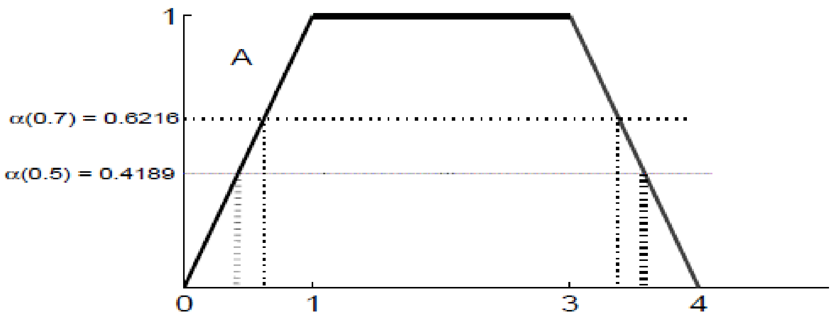

Fig. 7: Fuzzy number A from Example 8.

Up to now in this section we have introduced a new type of the percentiles and interval approximations of a fuzzy number, and some their properties are mentioned. Also, we will present another definition of the percentiles of a fuzzy number as follows:

Let $A \in F$ be a fuzzy number with the membership function $A(x)$ and

$$S_{P_\gamma^A}(A) = \int_a^{P_\gamma^A} A(x)dx, \tag{25}$$

Where $\gamma \in (0,1)$ and $P_\gamma^A \in suppA = [a,d]$. If $P_\gamma^A = a$ then $SP_\gamma^A\ (A) = 0$, and if $P_\gamma^A = d$ then $S_{P_\gamma^A}(A) = S(A) = Card(a)(see\ Fig.8)$.

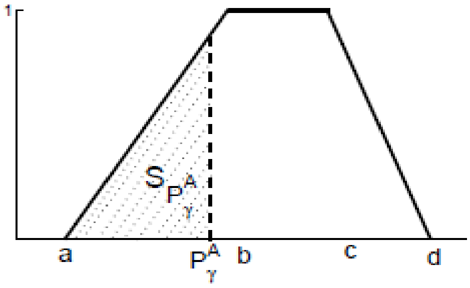

Fig. 8: Fuzzy number A and its γth percentile.

Definition 7:

Let $A \in F$. For any $\gamma \in (0,1)$ the value P_γ^A is called the $100\gamma th$ percentile of *A* with respect to the cardinality of *A* if

$$\frac{S_{P_\gamma^A}(A)}{S(A)} = \gamma. \tag{26}$$

In practice, this definition is more simple and interesting than Definition of Bodjanova (2005); especially, it is consistence with the definition of the percentiles in the probability and statistics contexts. We introduce an application of the percentiles (P_γ^A) by the following definition.

Definition 8:

Let $A \in F$. For any $\gamma \in (0,0.5)$, the interval

$$PI_{1-2\gamma}^A = [P_\gamma^A, P_{1-\gamma}^A], \tag{27}$$

called the interval $100(1-2\gamma)\%$ of the fuzzy number A.

Assume $\alpha_*(\gamma) = \min\left(A(P_\gamma^A), A(P_{1-\gamma}^A)\right), \gamma < 0.5$, then we define the new interval of A as follows:

$$CI_{\alpha*(\gamma)}^A = [a_1(\alpha_*(\gamma)), a_2(\alpha_*(\gamma))]. \tag{28}$$

The above intervals (26) and (27) showed in Fig.9. We can apply these fuzzy intervals for solving some the economical and financial problems, and transforming fuzzy matrices to interval matrices in the fuzzy analytic hierarchy process (FAHP).

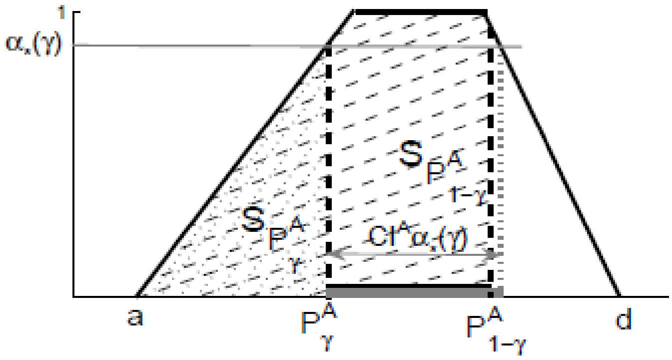

Fig. 9: Fuzzy number A and its interval approximations.

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Example 9:

Let A be a fuzzy number with the following membership function

$$A(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x, \mu \in R, \sigma > 0. \quad (29)$$

This function is called as a Gaussian membership function with a maximum value of 1, x is the independent variable on the universe, μ is the position of the peak relative to the universe, and $\sigma > 0$ standard deviation. Then for any $\alpha \in [0,1]$ we have

$$[A]_\alpha = [\mu - \sigma\sqrt{+2\ln(\alpha)}, \mu + \sigma\sqrt{-2\ln(\alpha)}].$$

We obtain the percentiles of fuzzy number A as follows

$$\int_{-\infty}^{P_\gamma^A} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \gamma \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \gamma\sqrt{2\pi}\sigma,$$

So,

$$\int_{-\infty}^{P_\gamma^A} \frac{1}{\sqrt{2\pi\sigma}} e^{(-\frac{(x-\mu)^2}{2\sigma^2}} dx = \gamma.$$

Consequently,

$$\emptyset = (\frac{P_\gamma^A - \mu}{\sigma}) = \gamma,$$

Therefore,

$$P_\gamma^A = \mu + \sigma\emptyset^{-1}(\gamma);\ \gamma \in (0,1),$$

Where $\emptyset(.)$ is the standard normal distribution function. For instance:

$$\mathrm{P}_{0.1}^{\mathrm{A}} = \mu + \sigma\emptyset^{-1}(0.1) = \mu - 1.28\sigma,$$

$$\mathrm{P}_{0.9}^{\mathrm{A}} = \mu + \sigma\emptyset^{-1}(0.9) = \mu + 1.28\sigma.$$

The interval

$$PI_{(1-2\times0.1)}^A = [\mathrm{P}_{(0.1)}^{\mathrm{A}}, \mathrm{P}_{(1-0.1)}^{\mathrm{A}}] = [\mu - 1.28\sigma, \mu + 1.28\sigma].$$

is an interval 80% of fuzzy number A with respect to its cardinality. If A is a symmetric fuzzy number then

$$PI_{(1-2\times0.1)}^A = [\mathrm{P}_{(0.1)}^{\mathrm{A}}, 2\mu - \mathrm{P}_{(0.1)}^{\mathrm{A}}] =$$

$$[\mu - 1.28\sigma, 2\mu - (\mu - 1.28\sigma)] = [\mu - 1.28\sigma, \mu + 1.28\sigma].$$

Furthermore,

$$CI_{\alpha*(\gamma)}^A = \alpha_*(\gamma) = \min(\mathrm{A}(\mathrm{P}_{(0.1)}^{\mathrm{A}}), \mathrm{A}(\mathrm{P}_{(0.9)}^{\mathrm{A}})) = \min(0.44, 0.44) = 0.44,$$

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$$CI_{(0.44)}^{A} = [\mu - 1.28\sigma, \mu + 1.28\sigma],$$

and this interval shows that 44th percentile (alpha-cut) of A in terms of the height of A is constructed an interval 80% of fuzzy number A with respect to its cardinality. Also we obtain $\alpha(0.65) = 0.44$,this means that $\frac{S_{\alpha(0.65)}(A)}{S(A)} = 0.65$. In other words, 65th percentile of A with respect to cardinality A as the horizontal is corresponding to 44th percentile in terms of height of $A\big(\alpha(\gamma) \in (0, hgtA)\big)$. Note that in this example we cannot compute the $100\gamma th$ percentile alpha-cut of A with respect to the width of A that introduced by Bodjanova (2006). Because the interval $I = [\mu, \mu]$ yields that $\frac{S_{(\mu,\mu)}(A)}{S(A)} = 0$.

5. Mode values and interval-valued mode:

In this section, First the three mode values of a fuzzy number are recalled, then we introduce a new mode value and several interval-valued modes of a trapezoidal fuzzy number. These points and intervals from the support function of the fuzzy number A are point and interval estimations with utmost membership grade. Let A be a fuzzy number with the continuous membership function. The mode value of A is defined as

$$Mode(A) = \{x|A(x) \geq 1\}.$$

A fuzzy number may have one or more than mode values.

Let $A = (a, b, c, d)$ be a trapezoidal fuzzy number with continuous membership function *A (x)*, we study four mode values for A.

(1) The first mode value of A is: $FMo(A) = \ min\,\{x|A(x) \geq 1\} = b.$

(2) The last mode value of A is: $LMo(A) = \ mix\,\{x|A(x) \geq 1\} = c.$

(3) The third mode value of A is middle point of the interval $IMo(A) = [FMo(A), LMo(A)]$, this means that,

$$MMo(A) = \frac{FMo(A) + LMo(A)}{2} = \frac{b + c}{2}.$$

We can rewrite the above items as follows:

Let $supp(A) = \{x \ \in X|A(x) > 0\}$ and $Core(A) = \{x \ \in X|A(x) = sup_{x \in X} A(x)\}$, where *X* is universe set and restrict to a bounded subset of the real line e.g. [a, d]. Then for a fuzzy number A we have

$FMo(A) = minCore(A),\ LMo(A) = maxCore(A),$

$$MMo(A) = \frac{minCore(A) + maxCore(A)}{2}.$$

(4) By geometry analytic, we propose a new method to find the mode value of a trapezoidal fuzzy number (see Fig.10). Let A = (a, b, c, d) be a trapezoidal fuzzy number, and let $P_1 = (a, 0),\ P_2 = (c, 1),\ P_3 = (b, 1),\ P_4(d, 0)$ be four points on A. Also, suppose (x_{Mo}, y_{Mo}) be the intersection point of lines $\overline{P_1P_2}$ and $\overline{P_3P_4}$ (see Fig.10), then we get the lines

$\overline{P_1P_2} : y = \frac{x-a}{c-a}$ and $\overline{P_3P_4} : y = \frac{d-x}{d-b}$; therefore

$$(x_{Mo}, y_{Mo}) = \left(\frac{cd-ab}{(c+d)-(a+b)}, \frac{(c+d-a-b)-(cd-ab)}{(c-a)(c+d-a-b)}\right).$$

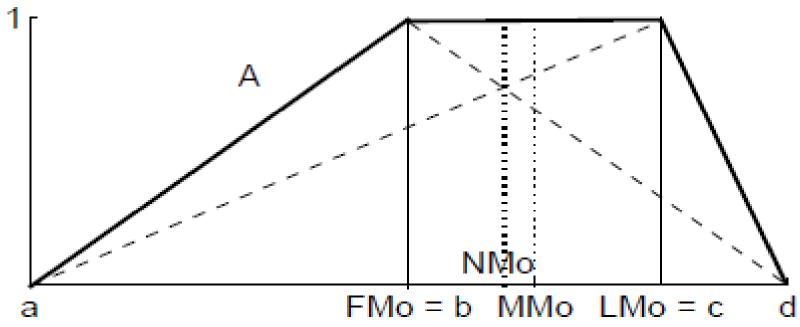

Fig. 10: Fuzzy number A and its modes.

Definition 9:

Let A = (a, b, c, d) be a trapezoidal fuzzy number. The new mode value of A is defined as

$$NMo(A) = \frac{cd-ab}{(c+d)-(a+b)} \tag{30}$$

the membership grade NMo(A) is 1, this means that A(NMo(A)) =1.

Theorem 3:

Let B = (a, b, c, d)$_n$. Then the new mode value of the fuzzy number B is $NMo(B) = \frac{cd-ab}{(c+d)-(a+b)}$ and $B\big(NMo(B)\big) = 1$.

Proof. The proof is simple.

This theorem shows that if *k* is the new mode value from the trapezoidal fuzzy number A = (a, b, c, d)$_n$, then *k* is the new mode value of fuzzy number B = (a, b, c, d)$_n$ too.

However, we can consider the four values for the mode of trapezoidal fuzzy number A, also there are the six interval-valued modes as:

$\mathrm{IM}_{o1}(A) = [FMo(A), LMo(A)], IM_{o2}(A) = [FMo(A), MMo(A)],$

$\mathrm{IM}_{o3}(A) = [MMo(A), LMo(A)], IM_{o4}(A) = [FMo(A), MMo(A)],$

$\mathrm{IM}_{o5}(A) = [NMo(A), LMo(A)],$

$\mathrm{IM}_{o6}(A) = \big[min(MMo(A), NMo(A)\big), max\,(MMo),$ NMo(A))].

The above intervals can be applied in the problems of interval and fuzzy equations solution that are of perennial interest, because of their direct relevance to practical modeling and optimization of real- world processes including finance, economy, mechanic and etc.

Example 10:

Suppose that a form produces a product that can be soled for a price of P per unit such that its total annual revenues equals TR = P.Q. The variable Q represents the number of units of the product produced and sold by the firm. Further suppose that the firm incurs two types of cost in its production process, fixed and variable. Fixed cost FC include overhead expenditures totaling per year. Variable cost VC incurred by the firm include raw materials and direct labor. Total variable costs are simple the product of variable cost per unit VC and total production output Q. The firm's profit function is defined as follows as total revenues minus the sum of fixed and variable costs:

$$\pi = P.Q - (FC + VC.Q)$$

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We can observe from the above equation that if the firm were to maintain an output level Q = 0, its profit would equal -FC due to its fixed costs. As the firm's output (and sales level) Q increases, profits π will increase. The firm's break-even production level is determined by solving for Q when profits π equal zero, and hence, the following expression can be used to determine the break-even production level when the firm has linear revenue and cost function:

$$Q^* = \frac{FC}{P - VC}.$$

Now let we have a case of uncertain for *P*,*Q* and *VC*. Therefore, we suppose that these variables are fuzzy intervals as

$$P = [P_1, P_2], Q = [Q_1, Q_2], VC = [VC_1, VC_2],$$

then the fuzzy firm's pro t function is defined as follows:

$$[\pi_1, \pi_2] = [P_1, P_2][Q_1, Q_2] - (FC + [VC_1, VC_2][Q_1, Q_2]),$$

so the break-even product level interval obtain as:

$$[Q_1^*, Q_2^*] = [\min\{\frac{FC}{P_1 - VC_1}, \frac{FC}{P_2 - VC_2}\}, \max\{\frac{FC}{P_2 - VC_1}, \frac{FC}{P_1 - VC_2}\}].$$

For instance, if P = [9, 11], VC = [5, 7] and FC = 1000$, then the break-even product level interval is

$[Q_1^*, Q_2^*] = [166.67., 500]$. is $[Q_1^*, Q_2^*] = [166.67., 500]$.

Note that we can apply other fuzzy intervals to obtain the break-even product level interval.

Definition 10:

(Nguyen and Wu 2006) Let U be the universe set, $X = \{x_1, x_2, \dots, x_k\}$a set of k-linguistic variables on U, and $\{FI_i = [a_i, b_i]; a_i, b_i \in R, i = 1,2, \dots, n\}$ be a sequence of random fuzzy sample on U. For each sample FI_i, If there is an interval [c, d] which is covered by certain samples, we call these samples as a cluster. Let MS be the set of clusters which contains the maximum number of sample. Then the fuzzy mode FIMo is defined as:

$$FIMo = [a, b] = \{\cap [a_i, b_i] | [a_i, b_i] \subset MS\}. \quad (31)$$

If [a, b] does not exist, then this fuzzy sample does not have the fuzzy mode.

Example 11:

Let U be the universe set which can be seen as an interval set,

$$FI_1 = [.5, 1.5], FI_1 = [1, 1.5], FI_1 = [.5, 2.5], FI_1 = [.25, 1], FI_1 = [.5, 2]$$

be the fuzzy intervals for the hours of exercise per day. Then, the fuzzy mode is $FIMo = [a, b] = [0.5, 1]$.
In the Table 2, the point and interval approximations of the fuzzy number $A = (a, b, c, d)_n$ with utmost membership grade are summarized.

Table 2: Point and interval approximations with utmost membership grade.

Point approximations		Interval approximations	
Previously	New	Previously	New
FMo, LMo, MMo	NMo	IMo1, IMo2, IMo3,	IMo4, IMo5, IMo6

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6. Approximations comparison:

In this section, we compare some of the point and interval approximations of fuzzy numbers with each other.

Theorem 3:

Let A be a symmetric fuzzy number with $[A]_\alpha = [k - s(\alpha), k + s(\alpha)]$, and λ_1, λ_2 be real numbers. Then

$$E(A) = M(A) = Me(A) = Mo(A) = k. \tag{32}$$

$$E(\lambda_1 A + \lambda_2) = M(\lambda_1 A + \lambda_2) = Me(\lambda_1 A + \lambda_2) = Mo(\lambda_1 A + \lambda_2) = \lambda_1 k + \lambda_2. \tag{33}$$

Note that $k = \frac{a_1(\alpha) + a_2(\alpha)}{2}$, for all $\alpha \in [0,1]$.

Theorem 4:

Let $A \in F_n^T(R)$, and $f(\alpha)$ be a monotone weighting function. Then

$$\lim_{n \to \infty} C_d^f(A) = [b, c] = C_1(A)$$

Proof:

Suppose that $f(\alpha)$ be a monotone weighting function and $A \in F_n^T(R)$ with $\alpha - \mathrm{cut}$ set $[A]_\gamma = \left[a + (b-a)\alpha^{\frac{1}{n}}, d - (d-c)\alpha^{\frac{1}{n}}\right]$. Then $C_d^f(A) = [C_L, C_U]$, where

$$C_L = \int_0^1 f(\alpha)\left(a + (b-a)\alpha^{\frac{1}{n}}\right)d\alpha = a + (b-a)\int_0^1 f(\alpha)\alpha^{\frac{1}{n}}d\alpha,$$

$$C_U = \int_0^1 f(\alpha)\left(d - (d-c)\alpha^{\frac{1}{n}}\right)d\alpha = d - (d-c)\int_0^1 f(\alpha)\alpha^{\frac{1}{n}}d\alpha,$$

Since the function $f(\alpha)$ is non-negative, monotone and it does not depend on *n*, furthermore

$$\lim_{n \to \infty} \alpha^{\frac{1}{n}} = 1, \int_0^1 f(\alpha) = 1.$$

Hence, by using theorems in the mathematical analysis we have

$$\lim_{n \to \infty} C_L = a + (b-a) = b, \qquad \lim_{n \to \infty} C_U = d - (d-c) = c,$$

so

$$\lim_{n \to \infty} C_d^f(A) = [b, c] = C_1(A).$$

Theorem 5:

Let$A \in F_n^T(R)$. Then

$$IMo(A) = C_1(A) = [b, c] \subseteq IM(A) \subseteq EI(A). \tag{34}$$

Theorem 6:

Let $A \in F(R)$ and $\gamma \in (0,0.5)$. Then

$$PI_{1-2\gamma}^A = [P_\gamma^A, P_{1-\gamma}^A] \subseteq [a_1(\alpha_*(\gamma)), a_2(\alpha_*(\gamma))] = CI_{\alpha_*(\gamma)}^A. \tag{35}$$

Proof:

We consider the different cases.

Case 1:

If $P_\gamma^A, P_{1-\gamma}^A \leq b$, then $\alpha_*(\gamma) = A(P_\gamma^A)$, so

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$$a_1(\alpha_*(\gamma)) = P_\gamma^A, a_2(\alpha_*(\gamma)) \geq b \geq P_{1-\gamma}^A,$$

Consequently,

$$[P_\gamma^A, P_{1-\gamma}^A] \subseteq [a_1(\alpha_*(\gamma)), a_2(\alpha_*(\gamma))].$$

Case 2:

If $P_\gamma^A \leq b, b \leq P_{1-\gamma}^A \leq c$, then $\alpha_*(\gamma) = A(P_\gamma^A)$, so

$$a_1(\alpha_*(\gamma)) = P_\gamma^A, a_2(\alpha_*(\gamma)) \geq c \geq P_{1-\gamma}^A,$$

Consequently,

$$[P_\gamma^A, P_{1-\gamma}^A] \subseteq [a_1(\alpha_*(\gamma)), a_2(\alpha_*(\gamma))].$$

Case 3:

If $P_\gamma^A \leq b, c \leq P_{1-\gamma}^A \leq a_2(A(P_\gamma^A))$, then $\alpha_*(\gamma) = A(P_\gamma^A)$, so

$$a_1(\alpha_*(\gamma)) = P_\gamma^A, a_2(\alpha_*(\gamma)) \geq P_{1-\gamma}^A \geq c,$$

Consequently,

$$[P_\gamma^A, P_{1-\gamma}^A] \subseteq [a_1(\alpha_*(\gamma)), a_2(\alpha_*(\gamma))].$$

Case 4. If $P_\gamma^A \leq b, c \leq a_2(A(P_\gamma^A)) \leq P_{1-\gamma}^A$, then $\alpha_*(\gamma) = A(P_{1-\gamma}^A)$, so

$$a_2(\alpha_*(\gamma)) = P_{1-\gamma}^A, a_1(\alpha_*(\gamma)) \leq P_\gamma^A \leq b,$$

Consequently,

$$[P_\gamma^A, P_{1-\gamma}^A] \subseteq [a_1(\alpha_*(\gamma)), a_2(\alpha_*(\gamma))].$$

Similarly, one can check other cases.

Remark 4. If $A \in F$ be a symmetric fuzzy number with $[A]_\alpha = [k - s(\alpha), k + s(\alpha)]$ and $\gamma \in (0, 0.5)$, then

$$CI_{\alpha_*(\gamma)}^A = PI_{1-2\gamma}^A = [P_\gamma^A, 2k - P_\gamma^A]. \tag{36}$$

Note that our method (Fig.11 (3) and Fig.11(4)) to obtain the percentile approximations of fuzzy numbers are different from Bodjanova method (2006) (Fig.11(1) and Fig.11(2)), because

1. Our method is consistent with the probability and statistics contexts. As the percentile is a point below which a stated percentage of the observations lie.

2. The interval approximation in Fig.11(1) (Bodjanova 2006) and Fig.11(3) (our method) equals, but the methods are different.

3. In Fig.11(4) (our method), the interval is obtained based on the definition of the percentiles in the statistical methods, and we get only an interval approximation, but Fig.11(2). Bodjanova (2006) method says there is the interval $Ix = [x_1, x_2]$from the fuzzy number A that it is include $100\gamma\%th$ of the area under membership function of fuzzy number. This means that $\frac{S_{Ix}(A)}{S(A)} = \gamma$. Obviously, in our method the mathematical computing is more simple than Bodjanova method (2006).

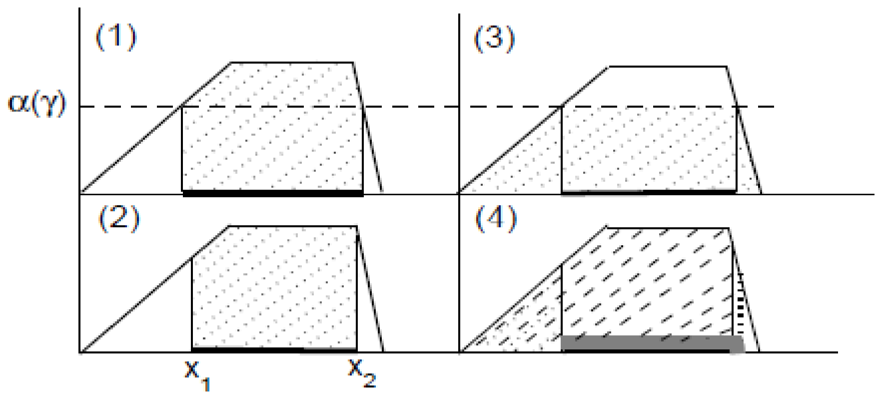

Fig.11: Fuzzy number A and interval approximations.

Note that the Fig.11 is also valid for other fuzzy numbers. We summarize the point and interval approximations of the fuzzy number A based on the percentiles in the Table 3.

Table 3: Point and Interval approximations.

Point approximations		Interval approximations	
Previously	New	Previously	New
Me	P_γ^A	IMe, $[a_1(\alpha(\gamma)), a_2(\alpha(\gamma))]$, $Ix = [x_1, x_2]$	$C_{\alpha(\gamma)}$, $PI_{1-2\gamma}$, $CI_{\alpha_*(\gamma)}$

7. The correlation coefficient between two intervals:

In this section, we introduce a type of correlation coefficient between two intervals such that its range is in [-1,1]. Correlation analysis is more concerned with the nature of the relationship between the two variables, with a central focus on the strength of that relationship. Thus, we state the following correlation measure.

Definition 12:

Let $I_1 = [a_1, a_2]$ and $I_2 = [b_1, b_2]$ be two close intervals, where

$a_1 \leq a_2, b_1 \leq b_2$

$$\rho(I_1, I_2) = \frac{a_1 b_1 + a_2 b_2}{\sqrt{(a_1^2 + a_1^2)(b_1^2 + b_1^2)}} \tag{37}$$

is called the correlation coefficient between I_1 and I_2.

Remark 5:

For all the close intervals I_1 and I_2 we have

(i) $\rho(I_1, I_2) = \rho(I_2, I_1)$.

(ii) If $I_1 = I_2$ then $\rho(I_1, I_2) = 1$.

(iii) If $I_1 = cI_2$ for some $c > 0$, then $\rho(I_1, I_2) = 1$.

(iv) $|\rho(I_1, I_2)| \leq 1$.

Now, let $[A]_\alpha = [b_1(\alpha), b_2(\alpha)]$ be two fuzz numbers, then for any $\alpha \in [0,1]$ we define

$$\rho_\alpha(A, B) = \frac{a_1(\alpha) b_1(\alpha) + a_2(\alpha) b_2(\alpha)}{\sqrt{(a_1^2(\alpha) + a_2^2(\alpha))(b_1^2(\alpha) + b_2^2(\alpha))}} \tag{38}$$

This correlation coefficient analysis the joint relationship of two fuzzy intervals can be examined with the aid of a measure of interdependence of the two intervals. Typical subject for correlation analysis are the association between height of father and height of son, between score in mathematics and score in statistics, etc. Also the correlation coefficient between fuzzy intervals ranges in [-1,1], which can correlate fuzzy concepts such big vs. heavy; beautiful vs. ugly, etc.

Example 12:

Consider the triangular fuzzy numbers $A = (0,0,1)$ and $B(0,1,1)$ (see Fig.12). We obtain

$\rho_0(A,B) = 1$, $\rho_{0.1}(A,B) = 0.955$, $\rho_{0.2}(A,B) = 0.981$, $\rho_{0.3}(A,B) = 0.958$, $\rho_{0.4}(A,B) = 0.928$, $\rho_{0.5}(A,B) = 0.894$, $\rho_{0.6}(A,B) = 0.857$, $\rho_{0.7}(A,B) = 0.819$, $\rho_{0.8}(A,B) = 0.781$, $\rho_{0.9}(A,B) = 0.743$,

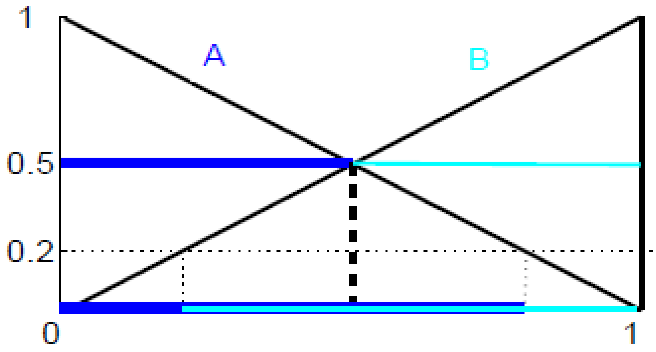

Fig. 12: Fuzzy numbers A, B and intervals approximations.

The above example shows that the correlation value of between two fuzzy numbers A and B decrease with increasing of α. We also can compute the correlation coefficients between the fuzzy intervals that introduced in before sections. These correlation coefficients show not only the degree of the relationship between the interval-valued intuitionistic fuzzy sets, but also the fact that these two sets are positively or negatively related, which are better than the correlation coefficients that lie in [0,1].

Example 13:

Consider the following interval approximations from Example 4.

$EI\left(\tilde{\tilde{\theta}}\right) = [1.5, 2.5], IM\left(\tilde{\tilde{\theta}}\right) = [2,2],$

$EI^*\left(\tilde{\tilde{\theta}}\right) = [1,3], IM^*\left(\tilde{\tilde{\theta}}\right) = [1.333, 2.667].$

We get $\rho(EI, IM) = 0.8701$ and $\rho(EI^*, IM^*) = 0.9900$,

This example shows that the relations (15),(16) have more information than relation (11); intuitively.

Example 15:

Consider the five triangular fuzzy numbers are defined with the corresponding membership functions as shown in Fig.13. We get the correlation coefficients matrix for the level $\alpha = 0$ as follows:

$$\rho_0 = \begin{array}{c} \tilde{1} \\ \tilde{3} \\ \tilde{5} \\ \tilde{7} \\ \tilde{9} \end{array} \begin{pmatrix} 1 & 0.9923 & 0.9965 & 0.9828 & 0.9701 \\ & 1 & 0.9785 & 0.9524 & 0.9326 \\ & & 1 & 0.9948 & 0.9869 \\ & & & 1 & 0.9982 \\ & & & & 1 \end{pmatrix}$$

Similarly, one can obtain the correlation coefficients matrices for the different levels and the interval approximations. Notice: Triangular fuzzy numbers $(\tilde{1}, \tilde{3}, \tilde{5}, \tilde{7}, \tilde{9})$can be used to indicate the relative strength of each pair of elements in the fuzzy AHP. However, these matrices can be applied to favorite judgment in the fuzzy AHP.

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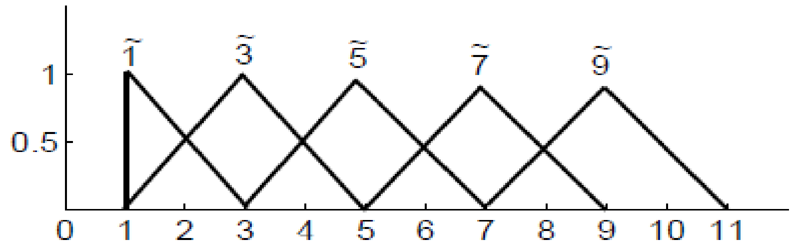

Fig. 13: The membership functions of triangular fuzzy numbers $\tilde{1}, \tilde{3}, \tilde{5}, \tilde{7}, \tilde{9}$.

In the end, we mentioned several applications of the point and interval approximations of fuzzy numbers so that one can research on them by using these operators in future, and the new methodologies propose for the fuzzy problems.

1. The point and interval operators can be used to transform the fuzzy comparison matrices to the point and interval comparison matrices in the fuzzy AHP, and the fuzzy data envelopment analysis (FDEA).

2. The percentiles of fuzzy numbers can be used to find the skewness and kurtusis of fuzzy numbers for the better recognition of the patterns in the fuzzy models. For instance, we can find the skewness and kurtusis of fuzzy numbers in the times series models.

3. Some of these operators can be used for ranking fuzzy numbers (see Goetschel and . Voxman (1986)).

4. Index numbers are measure of relative change which are usually based on rations. These operators can be used for transforming of the price indices (Laspeyres, Fisher, Marshal, etc.) to the fuzzy price indices that play an important role in the economical problems.

5. The interval operators may be used to estimate of the parameters of statistical models in a fuzzy environment.

8. Conclusion:

In this paper, we have introduced the weighted interval approximation of a fuzzy number, which is an extension of the works of Grzegorzewski. The new fuzzy interval-valued means, and the interval-valued modes of a fuzzy number are proposed. Furthermore, based on the percentiles of fuzzy numbers the new point and interval approximations of a fuzzy number are presented. Also a type of the correlation coefficient is mentioned for comparing interval approximations. These crisp approximately operators of fuzzy numbers are simple, suitable and play an important role in the fuzzy problems. In future can research on the further applications of these operators.

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